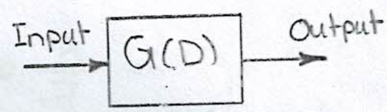
1.1 Representation of Control System Components.

To investigate the performance of control systems, it's necessary to obtain the mathmatical relationship relating controlled variable to the reference input for both of the following systems :

I. Open-Loop Control System:

A system in which the output has no effect on the control action (the output neither measured non feedback for comparison with the input). The transfor function of such control system is :



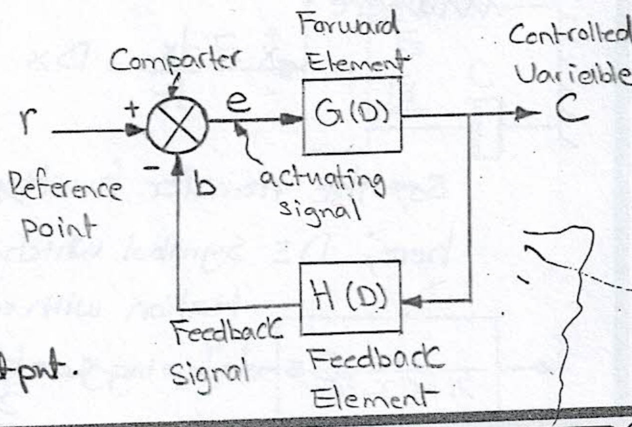
Open-Loop System

$$\boxed{\frac{\text{Output}}{\text{Input}} = G(D)}$$

II. Closed-Loop Control System:

A system that maintain a relationship between the output and the reference input by comparing them and using the difference as means of control.

The transfer function could be obtained by writing the mathematical equations describing the operation of each component between input and output.



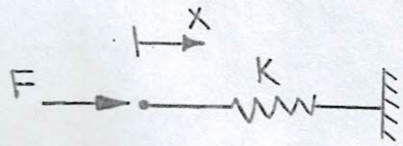
1.2 Control Systems

1.2.1 Mechanical Components.

A) Spring:

The relation of the spring is:

$$F = K.x$$



where:

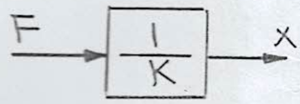
$x \equiv$ displacement (m)

$K \equiv$ stiffness (N/m)

however, system transfer function $= \dfrac{\text{Output}}{\text{Input}}$

$$\therefore \frac{X}{F} = \frac{1}{K}$$

$\therefore$ the spring system above can be represented in a block diagram form as:



Block Diagram Form

here;

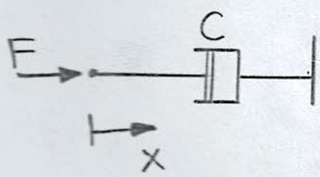
$F \equiv$ Input $\quad X \equiv$ Output

$\dfrac{1}{K} \equiv$ system transfer function.

B) Viscous Damping

The relationship of the damper is:

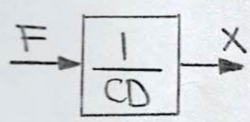
$$F = C\dot{x}$$



where:

$$\dot{x} = \frac{dx}{dt} = Dx$$

So, the transfer function is $\dfrac{X}{F} = \dfrac{1}{CD}$



Block Diagram Form

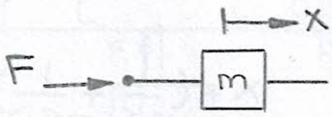
here; $D \equiv$ Symbol which indicates differentiation with respect to time.

$C \equiv$ damping coefficient (N.s/m)

C. Mass

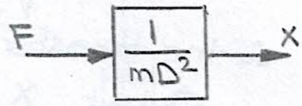
The relationship of mass is :

$$F = m.\ddot{x} \Rightarrow F = m.D^2.x$$



$\therefore$ the transfer function is :

$$\frac{X}{F} = \frac{1}{mD^2}$$



m: mass (Kg)

Block Diagram

I. Mechanical Elements Connections

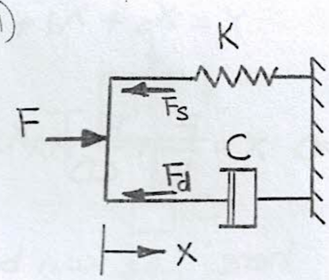
Case 1 : ~~Series~~ Connection (Parallel)

$$F = F_{spring} + F_{damping}$$

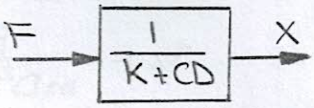
$$F = K.x + C\dot{x}$$

$$= K.x + CDx$$

$$F = (K + CD)\,x$$



$$\Rightarrow \therefore \frac{\text{Output}}{\text{Input}} = \frac{X}{F} = \frac{1}{K+CD} \Rightarrow$$



Case 2 : Second order System

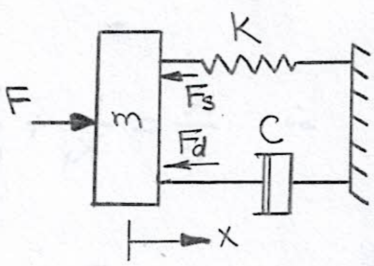
$$\sum \text{Forces} = m.\ddot{x}$$

$$\therefore F - F_s - F_d = m\ddot{x}$$

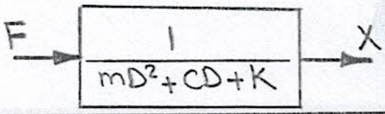
$$F - Kx - C\dot{x} = m\ddot{x}$$

$$F = mD^2x + CDx + Kx$$

$$F = (mD^2 + CD + K)\,x$$



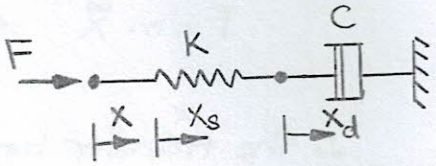
$$\therefore \frac{\text{Output}}{\text{Input}} = \frac{X}{F} = \frac{1}{mD^2+CD+K} \Rightarrow$$



Case 3 : ~~Parallel~~ Connection (Series)

$$X = X_s + X_d$$

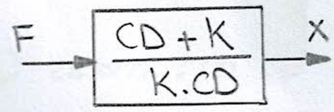
$$X = F/K + F/CD$$



$$X = \left(\frac{1}{K} + \frac{1}{CD}\right) F$$

∴ the transfer function is

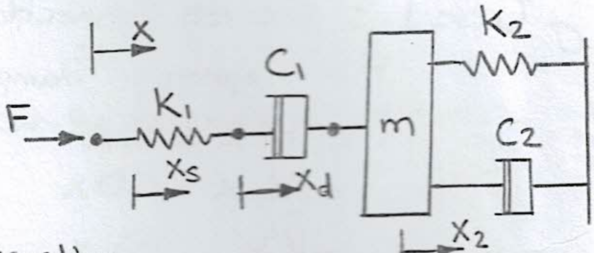
$$\frac{\text{Output}}{\text{Input}} = \frac{X}{F} = \frac{1}{K} + \frac{1}{CD} = \frac{CD + K}{K.CD}$$



Case 4 :

$$X = X_s + X_d + X_2 \qquad (1)$$

$$\Rightarrow X = \frac{F}{K_1} + \frac{F}{CD} + X_2 \qquad (2)$$



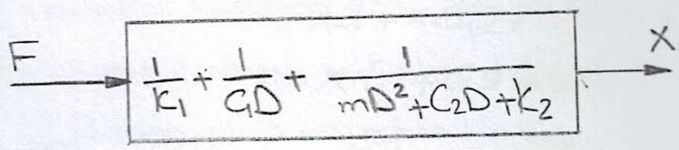
here; X_2 can be found according to case (2) as :

$$X_2 = \frac{F}{mD^2 + C_2 D + K_2} \qquad (3)$$

∴ from equations (2) and (3), we can obtain ;

$$X = \frac{F}{K_1} + \frac{F}{C_1 D} + \frac{F}{mD^2 + C_2 D + K_2}$$

$$\therefore \quad \frac{X}{F} = \frac{1}{K_1} + \frac{1}{C_1 D} + \frac{1}{mD^2 + C_2 D + K_2}$$

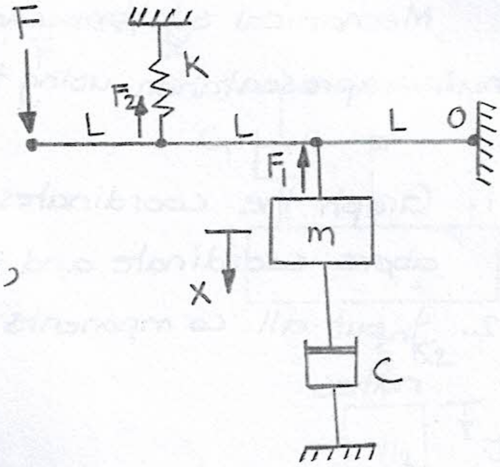


Case 5 :

$\sum M_o = 0$

$F(3L) = F_2(2L) + F_1(L)$

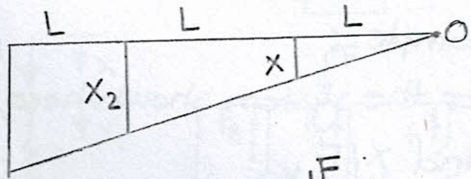
$\therefore \; 3F = 2F_2 + F_1 \qquad (1)$



from similarity of triangles,
we can obtain:

$$\frac{X}{L} = \frac{X_2}{2L}$$

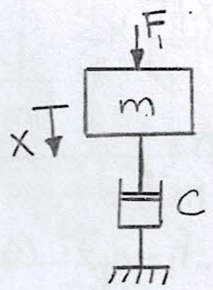
$\therefore \; X_2 = 2X \qquad (2)$



$F_1 - F_d = m\ddot{X}$

$F_1 = mD^2X + CDX \qquad (3)$

$F_2 = KX_2 = 2KX \qquad (4)$



from equations (1), (3) and (4)

$\Rightarrow \; 3F = mD^2X + CDX + 2(2KX)$

$$\therefore \; \frac{X}{F} = \frac{3}{mD^2 + CD + 4K}$$

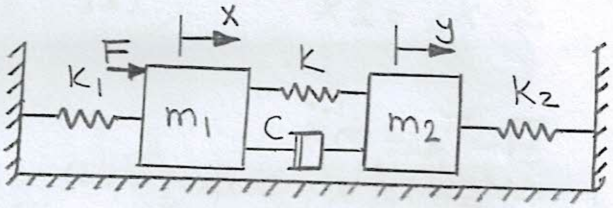
II. Grounded - Chair Representation.

Mechanical systems can be represented by grounded chair representation using the steps as :

1. Graph the coordinates and put the forces effect at the above coordinate and the ground below.
2. Input all components, and array it with the coordinates.

Example :

For the system shown here, find X/F.



Solution :

$\Sigma F = m\ddot{x}$

$F - f_s - f_1 = m_1\ddot{x}$ (1)

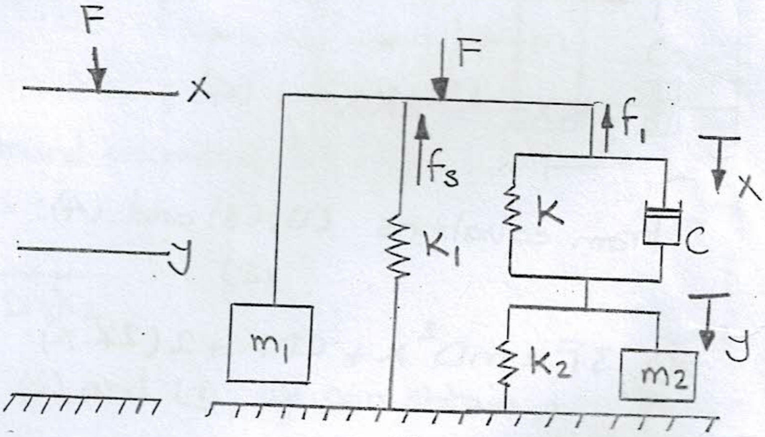


Figure (1)

from figure (2),

$X = X_1 + y$

$$X = \frac{f_1}{CD+K} + y$$

OR

$$X = \frac{f_1}{CD+K} + \frac{f_1}{m_2D^2+K_2}$$

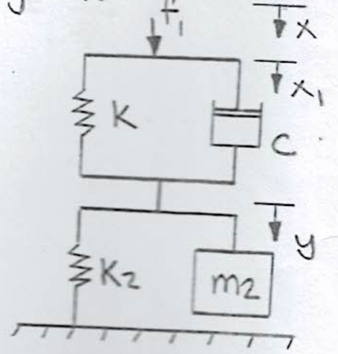


Figure (2)

$$\therefore f_1 = \frac{X}{\frac{1}{CD+K} + \frac{1}{m_2D^2+K_2}} \quad (2)$$

now, sub. (2) in (1) $\Rightarrow$

$$F = \left(m_1D^2 + K_1 + \frac{1}{\frac{1}{CD+K} + \frac{1}{m_2D^2+K_2}}\right) X$$

H.W. : Find $\frac{y}{X}$ & $\frac{y}{F}$

Example :

For the system shown, find $\frac{x}{F}$, $\frac{y}{F}$ and $\frac{y}{x}$

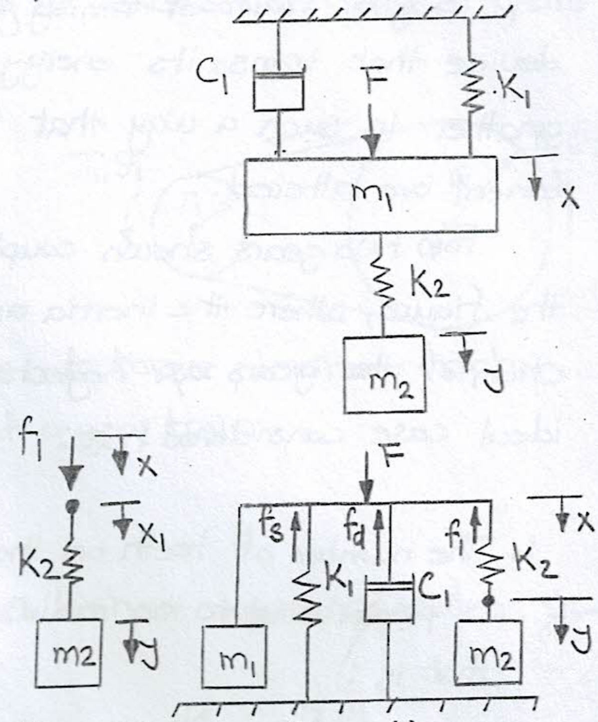


Figure (2)   Figure (1)

Solution:

$\Sigma F = m_1 . \ddot{x}$

$F - f_s - f_d - f_1 = m_1 \ddot{x}$ (1)

from figure (2); we get :

$x = x_1 + y$

$x = \frac{f_1}{K_2} + y$ (2)

and;

$f_1 = m_2 . \ddot{y}$

$f_1 = m_2 . D^2 . y$ (3)

from equations (2) and (3),

$K_2(x - y) = m_2 D^2 . y$ OR $x = \frac{f_1}{K_2} + \frac{f_1}{m_2 D^2}$ (4)

$\Rightarrow K_2 x = (m_2 D^2 + K_2) y$

$\therefore \frac{y}{x} = \frac{K_2}{m_2 D^2 + K_2}$ (5) ✓

Sub. equation (4) in equation (1) $\Rightarrow$

$$F - K_1 x - C_1 D x - \frac{x}{\frac{1}{K_2} + \frac{1}{m_2 D^2}} = m_1 . D^2 . x$$

$$\therefore F = \left(m_1 D^2 + C_1 D + K + \frac{1}{\frac{1}{K_2} + \frac{1}{m_2 D^2}} \right) x \quad (6) \checkmark$$

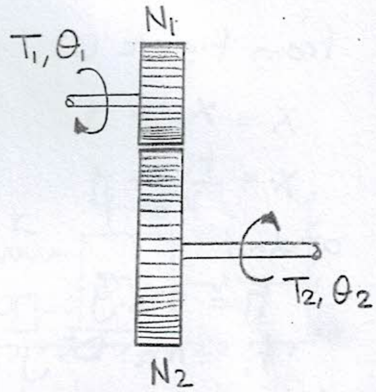
now, sub. equation (8) in equation (6) $\Rightarrow$

$$F = \left(m_1 D^2 + C_1 D + K_1 + \frac{1}{\frac{1}{K_2} + \frac{1}{m_2 D^2}} \right) \left(\frac{m_2 D^2 + K_2}{K_2} \right) y \quad \checkmark$$

III. Gear Trains and Timing Belts.

A gear train or timing belt over pullys is a mechanical device that transmits energy from one part of a system to another in such a way that force, torque, speed and displacement are altered.

For two gears shown coupled in the figure, where the inertia and friction of the gears are neglected in the ideal case considered here:



1. The number of teeth on the gear is proportional to the radius of gears, that is :

$$\frac{r_1}{r_2} = \frac{N_1}{N_2} \qquad (1)$$

2. The linear distance traversed along the surface of each gear is same. Therefore ;

$$r_1 \theta_1 = r_2 \theta_2$$

OR

$$\frac{r_1}{r_2} = \frac{\theta_2}{\theta_1} \qquad (2)$$

3. The work done by one gear is same as that of the other

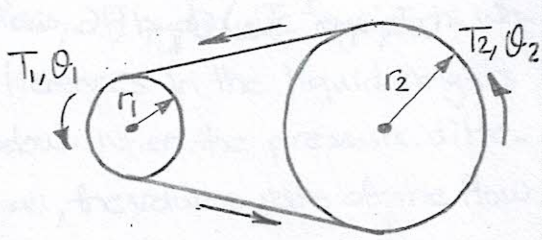
$$T_1 . \theta_1 = T_2 . \theta_2 \qquad (3)$$

from equations (1), (2) and (3) with the angular velocities of the two gears ω_1 and ω_2 lead to ;

$$\frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} = \frac{N_1}{N_2} = \frac{\omega_2}{\omega_1} = \frac{r_1}{r_2} = n \qquad (4)$$

For timing belts and chain drives serve the same purposes as the gear train except that they allow the transfer of energy over a longer distance with using an excessive number of gears as shown.

Assuming that there is no slippage between the belt and the pulleys, the equation (4) can be applied to this case.



The reflection and transmittance of torque, inertia, friction and so on, is similar to that of a gear train.

Example: In practice, two gears do have inertia coupled as shown here, where:

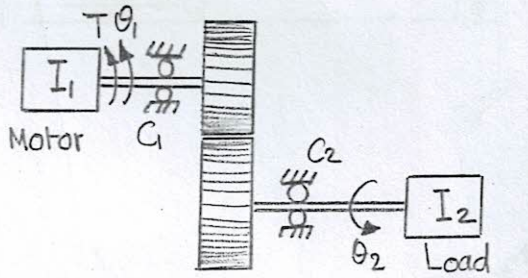
T: applied torque by motor

θ_1, θ_2: angular displacements.

I_1, I_2: mass moment of inertia

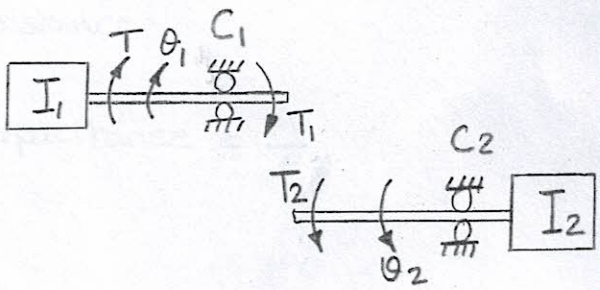
C_1, C_2: coefficients of damping

K_1, K_2: torsional stiffness



$n = \frac{N_1}{N_2}$ = gear ratio

or $n = \frac{\omega_2}{\omega_1}$



Balancing the torque on the motor and load shafts are:

$$I_1\ddot{\theta}_1 + C_1\dot{\theta}_1 + K_1\theta_1 = T - T_1$$
$$I_2\ddot{\theta}_2 + C_2\dot{\theta}_2 + K_2\theta_2 = T_2$$

$$\because \quad T_1 \cdot \dot{\theta}_1 = T_2 \cdot \dot{\theta}_2 \quad \Rightarrow \quad \frac{T_1}{T_2} = \frac{\dot{\theta}_2}{\dot{\theta}_1} = n$$

$$\therefore\ I_1 \ddot{\theta}_1 + C_1 \dot{\theta}_1 + K_1 \theta_1 = T - n T_2 = T - n\left(I_2 \ddot{\theta}_2 + C_2 \dot{\theta}_2 + K_2 \theta_2\right)$$

by substituting $\theta_2 = n.\theta_1$ in the equation we get :

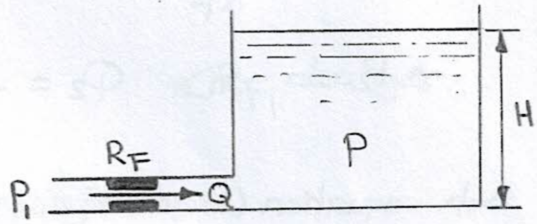
$$(I_1 + n^2 I_2)\ddot{\theta}_1 + (C_1 + n^2 C_2)\dot{\theta}_1 + (K_1 + n^2 K_2)\theta_1 = T$$

1.2.2 Hydraulic Systems.

Hydraulics is the study of incompressible liquids, and hydraulic devices use an incompressible liquids such as oil for their working medium. Liquid level systems consisting of storage tanks and connecting pipes are a class of hydraulic systems whose driving force is due to relative differences in the liquid heights in the tanks, as shown in the figure below. When the pressure difference across a flow restriction is small, the volume rate of the flow (Q) is proportional to the pressure drop ($P_1 - P$) across the restriction.

$$Q = \frac{P_1 - P}{R_F} \qquad (1)$$

$$Q = A.V = A.\dot{H} = ADH$$



$$\because P = \rho H$$

$$\therefore Q = \frac{A}{\rho} DP$$

$$\Rightarrow Q = C_F . DP \qquad (2)$$

where:

R_F ≡ equivalent liquid resistance

H ≡ the head of liquid

C_F ≡ equivalent liquid capacitance $= \frac{A}{\rho}$

ρ ≡ density

from equation (2) and (1), we get:

$$Q = \frac{P_1 - P}{R_F} = C_F . DP$$

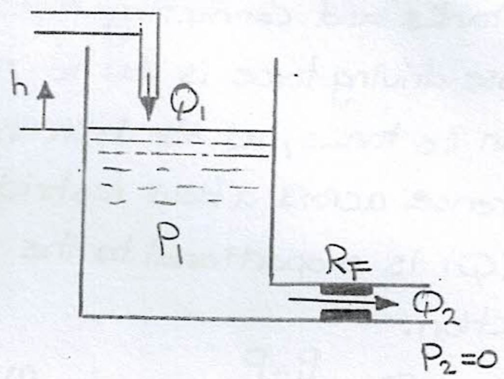
$$P_1 - P = R_F . C_F . DP \Rightarrow P_1 = P + R_F . C_F . DP$$

$$\therefore P = \frac{P_1}{1 + R_F . C_F . D} \qquad (3) \quad \text{OR} \quad P = \frac{P_1}{1 + TD}$$

Example: For the tank shown in the figure, obtain the transfer function relating the deviation in head ($h(t)$) as output to the deviation in flow (φ_1) as input.

Solution:

For the liquid balance in the tank;



$$\varphi_1 = \varphi_2 + A.\dot{h}$$

$$\varphi_1 = \varphi_2 + ADh \qquad (1)$$

$$\varphi_2 = \frac{P_1 - P_2}{R_F}$$

$$\because P_2 = 0 \Rightarrow \varphi_2 = \frac{P_1}{R_F} \qquad (2)$$

Sub. equation (2) in equation (1) $\Rightarrow \varphi_1 = \frac{P_1}{R_F} + ADh$

however, $P_1 = \rho g h$;

$$\therefore \varphi_1 = \left[\frac{\rho g}{R_F} + AD\right] h$$

$$\therefore \frac{h}{\varphi_1} = \frac{1}{\frac{\rho g}{R_F} + AD} = \frac{1}{\frac{\rho g}{R_F}\left(1 + \frac{A\rho g}{R_F}D\right)}$$

$$= \frac{\frac{R_F}{\rho g}}{\left(1 + \frac{A\rho g}{R_F}D\right)} = \frac{\frac{R_F}{\rho g}}{1 + \tau D}$$

where:

$$\tau = \frac{A\rho g}{R_F}$$

$$\varphi_1 \longrightarrow \boxed{\frac{R_F/\rho g}{1+\tau D}} \longrightarrow h(t)$$

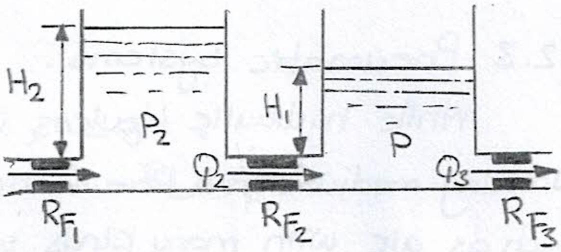
Example : Determine the equation for the pressure (P) as a function of the inlet (P_1) (P_2 should not appear in this equation).

Solution:

For the first tank :

$Q_1 = Q_2 + A\dot{H}_2$

$Q_1 = Q_2 + C_{F_1}.DP_2$



$\therefore Q_1 = \dfrac{P_1 - P_2}{R_{F_1}}$

$Q_2 = \dfrac{P_2 - P}{R_{F_2}}$

$\therefore C_{F_1} DP_2 = \dfrac{P_1 - P_2}{R_{F_1}} - \dfrac{P_2 - P}{R_{F_2}}$ $\quad * R_{F_1}$ to get :

$\Rightarrow R_{F_1}.C_{F_1}.DP_2 = P_1 - P_2 - \dfrac{R_{F_1}}{R_{F_2}}.P_2 + \dfrac{R_{F_1}}{R_{F_2}}.P$

$\therefore P_1 = \left(1 + R_{F_1}.C_{F_1}.D + \dfrac{R_{F_1}}{R_{F_2}}\right)P_2 - \dfrac{R_{F_1}}{R_{F_2}}.P \qquad (1)$

For the second tank :

$Q_2 = Q_3 + A\dot{H}_1$

$\quad = Q_3 + C_{F_2} DP$

$\therefore C_{F_2}.DP = Q_2 - Q_3$ $\qquad$ but $Q_2 = \dfrac{P_2 - P}{R_{F_2}}$ and $Q_3 = \dfrac{P - 0}{R_{F_3}}$

$\therefore C_{F_2}.DP = \dfrac{P_2 - P}{R_{F_2}} - \dfrac{P}{R_{F_3}}$ $\qquad \times R_{F_2}$ to get :

$R_{F_2}.C_{F_2}.DP = P_2 - P - \dfrac{R_{F_2}}{R_{F_3}}P$

$\therefore P_2 = \left(1 + R_{F_2}.C_{F_2}.D + \dfrac{R_{F_2}}{R_{F_3}}\right)P \qquad (2)$

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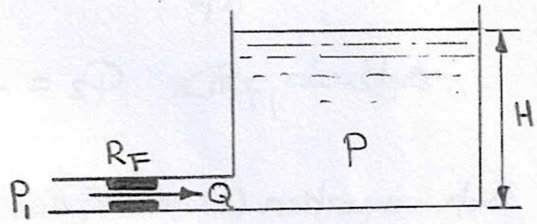

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$\rho \equiv$ density

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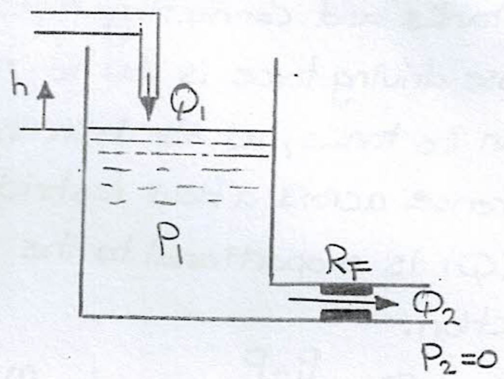
Solution:

For the liquid balance in the tank;



$$\varphi_1 = \varphi_2 + A.\dot{h}$$

$$\varphi_1 = \varphi_2 + ADh \qquad (1)$$

$$\varphi_2 = \frac{P_1 - P_2}{R_F}$$

$$\because P_2 = 0 \Rightarrow \varphi_2 = \frac{P_1}{R_F} \qquad (2)$$

Sub. equation (2) in equation (1) $\Rightarrow \varphi_1 = \frac{P_1}{R_F} + ADh$

however, $P_1 = \rho g h$;

$$\therefore \varphi_1 = \left[\frac{\rho g}{R_F} + AD\right] h$$

$$\therefore \frac{h}{\varphi_1} = \frac{1}{\frac{\rho g}{R_F} + AD} = \frac{1}{\frac{\rho g}{R_F}\left(1 + \frac{A\rho g}{R_F} D\right)}$$

$$= \frac{\frac{R_F}{\rho g}}{\left(1 + \frac{A\rho g}{R_F} D\right)} = \frac{\frac{R_F}{\rho g}}{1 + \tau D}$$

where :

$$\tau = \frac{A\rho g}{R_F}$$

$$\varphi_1 \longrightarrow \boxed{\frac{R_F/\rho g}{1+\tau D}} \longrightarrow h(t)$$

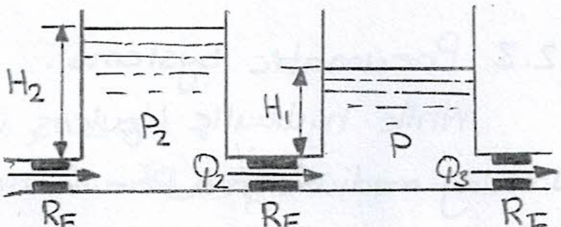
Example : Determine the equation for the pressure (P) as a function of the inlet (P_1) (P_2 should not appear in this equation).

Solution:

For the first tank :

$\varphi_1 = \varphi_2 + A\ddot{H}_2$

$\varphi_1 = \varphi_2 + C_{F_1}.DP_2$



$\therefore \varphi_1 = \dfrac{P_1 - P_2}{R_{F_1}}$

$\varphi_2 = \dfrac{P_2 - P}{R_{F_2}}$

$\therefore C_{F_1}\,DP_2 = \dfrac{P_1 - P_2}{R_{F_1}} - \dfrac{P_2 - P}{R_{F_2}}$ $\quad * R_{F_1}$ to get :

$\Rightarrow R_{F_1}.C_{F_1}.DP_2 = P_1 - P_2 - \dfrac{R_{F_1}}{R_{F_2}}.P_2 + \dfrac{R_{F_1}}{R_{F_2}}.P$

$\therefore P_1 = \left(1 + R_{F_1}.C_{F_1}.D + \dfrac{R_{F_1}}{R_{F_2}}\right)P_2 - \dfrac{R_{F_1}}{R_{F_2}}.P \qquad (1)$

For the second tank :

$\varphi_2 = \varphi_3 + A\ddot{H}_1$

$\quad = \varphi_3 + C_{F_2}\,DP$

$\therefore C_{F_2}.DP = \varphi_2 - \varphi_3$ $\quad$ but $\varphi_2 = \dfrac{P_2 - P}{R_{F_2}}$ and $\varphi_3 = \dfrac{P - 0}{R_{F_3}}$

$\therefore C_{F_2}.DP = \dfrac{P_2 - P}{R_{F_2}} - \dfrac{P}{R_{F_3}}$ $\quad \times R_{F_2}$ to get :

$R_{F_2}.C_{F_2}.DP = P_2 - P - \dfrac{R_{F_2}}{R_{F_3}}P$

$\therefore P_2 = \left(1 + R_{F_2}.C_{F_2}.D + \dfrac{R_{F_2}}{R_{F_3}}\right)P \qquad (2)$

sub. in equation (1) ;

$$P_1 = \left(1 + R_{F_1} \cdot C_{F_1} \cdot D + \frac{R_{F_1}}{R_{F_2}}\right)\left(1 + R_{F_2} \cdot C_{F_2} \cdot D + \frac{R_{F_2}}{R_{F_3}}\right)P - \frac{R_{F_1}}{R_{F_2}} P$$

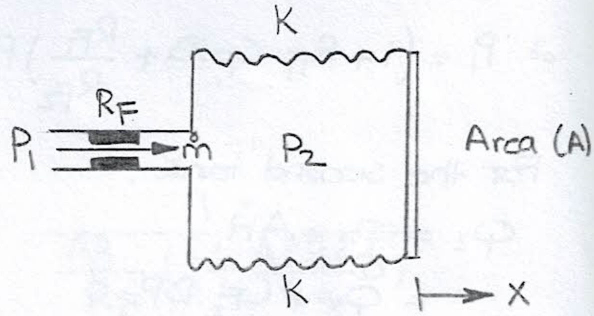
$$\therefore \frac{P}{P_1} = 1 / \left(1 + R_{F_1} \cdot C_{F_1} \cdot D + \frac{R_{F_1}}{R_{F_2}}\right)\left(1 + R_{F_2} \cdot C_{F_2} \cdot D + \frac{R_{F_2}}{R_{F_3}}\right) - \frac{R_{F_1}}{R_{F_2}}$$

1.2.3 Pneumatic Systems.

While hydraulic devices use an incompressible liquid, the working medium in a Pneumatic device is a compressible fluid, such as air with many kinds such as :

I. Pneumatic Bellow.

It is an expandable chamber, where the elasticity of the of the walls is represented by a spring, the change in pressure causes a displacement from equilibrium of the plane as shown in the figure. For small pressure difference, the mass rate of flow ($\dot{m}$) through a restriction is proportional to the pressure difference ($P_1 - P_2$), so :



$$\dot{m} = \frac{P_1 - P_2}{R_F} \qquad (1)$$

$$\because PV = m.R.T \Rightarrow m = \frac{PV}{RT} \qquad \text{and} \qquad \dot{m} = \frac{dm}{dt}$$

$$\Rightarrow \therefore \frac{dm}{dt} = \frac{V}{RT} \cdot \frac{dP}{dt} = C_F . DP$$

$$\therefore \dot{m} = C_F . DP_2$$

where :

$R_F \equiv$ the equivalent fluid resistance

$C_F \equiv \frac{V}{RT}$ equivalent fluid capacitance

for force balance of below ;

$P_2 . A = K . X$ (3)

from equations (1), (2) and (3), we get ;

$P_1 - P_2 = R_F . C_F . D P_2$ but $P_2 = \frac{K}{A} X$

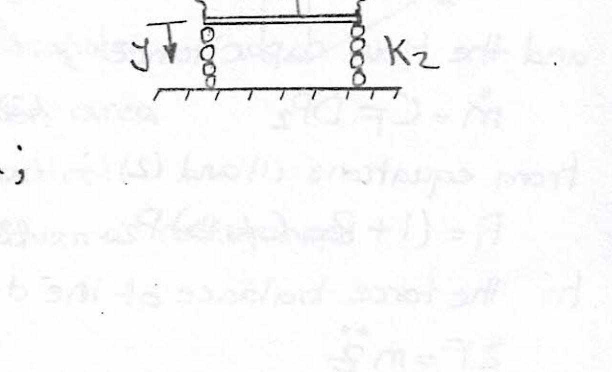
$\therefore X = \frac{A}{K(1 + R_F . C_F . D)} . P_1$

II. Pneumatic Flapper Valve.

The flapper valve consist of nozzel and lever, with a constant supply pressure (P_2) in chamber, controlled by the piston (X) of the flopper. Therefore, small changes in input motion (X) causes larg changes in the controlled pressure (P_2), as shown;

For the flopper motion without ($\dot{m}$) :

$P_2 = f(X)$



$P_2 \propto \frac{1}{X} \Rightarrow P_2 = -C_1 . X$ (1)

forces balance gives;

$P_2 . A_2 = K_2 . y$ (2)

$\Rightarrow P_2 = K_2 . y / A_2$

sub. in equation (1) results in;

$\frac{K_2}{A_2} . y = -C_1 . X$

$\therefore \frac{y}{X} = -\frac{C_1 . A_2}{K_2}$

however, for ($\dot{m}$) balance the equations are :

$$\dot{m}_1 = f(P_2) \Rightarrow \dot{m}_1 = -C_1 . P_2 \qquad (a)$$

and

$$\dot{m}_o = f(x, P_2) \Rightarrow \dot{m}_o = C_2 . X + C_3 . P_2 \qquad (b)$$

$$\dot{m}_1 - \dot{m}_o = A_2 . Dy \qquad (c)$$

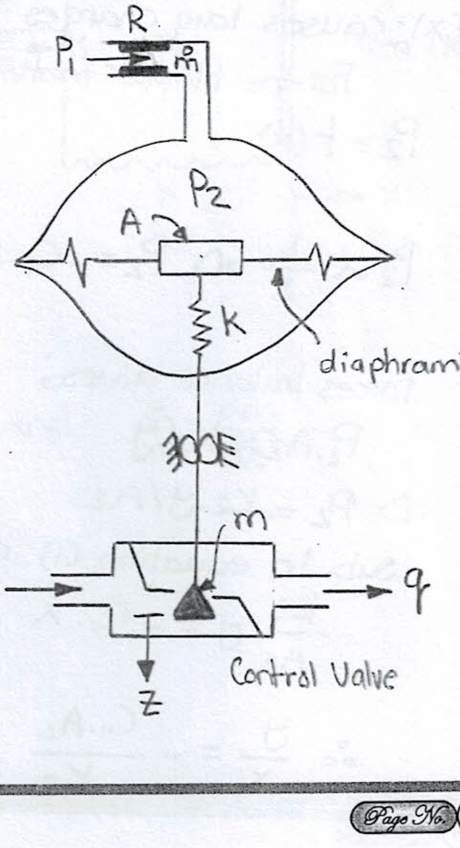
$$P_2 . A_2 = K_2 . y \qquad (d)$$

give $\Rightarrow \dfrac{y}{x} = -\dfrac{A_2 . C}{K_2(1+TD)}$

where $T = \dfrac{A_2^2}{K_2(C_1+C_3)}$ and $C = \dfrac{C_2}{C_1+C_3}$

III. Pneumatic Diaphram.

A force type Pneumatic controller operates only on pressure signals, and therefore it is necessary to convert the reference input and controlled variable to corresponding pressure. An example of pneumatic diaphram can be seen in the figure with mass flow rate ($\dot{m}$) of air flowing into chamber can be given as :



$$\dot{m} = \frac{P_1 - P_2}{R} \qquad (1)$$

and the flow capacitance ;

$$\dot{m} = C_F . DP_2 \qquad (2)$$

from equations (1) and (2) ;

$$P_1 = (1 + R_F . C_F . D) P_2 \qquad (3)$$

for the force balance of the diaphram;

$$\Sigma F = m\ddot{z}$$

$$\Rightarrow P_2.A = (mD^2 + CD + K)\,Z \qquad (4)$$

from equations (3) and (4), we get;

$$P_1 = (1 + R_F.C_F.D)\,\frac{mD^2 + CD + K}{A}.\,Z \qquad (5)$$

the flow rate through the control value is given by;

$$q = f(Z) \Rightarrow q = C_1.Z \qquad (6)$$

Sub. equation (6) in equation (5) $\Rightarrow$

$$P_1 = (1 + R_F.C_F.D)\,\frac{mD^2 + CD + K}{A}.\frac{q}{C_1}$$

$$\therefore\ q = \frac{AC_1}{(mD^2 + CD + K)(1 + R_F.C_F.D)}.\,P_1$$

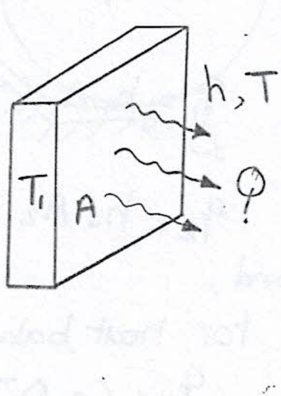
1.2.4 Thermal Systems.

It is connection with the system to be controlled, such as those found in chemical processes, power plants and heating-air conditioning of buildings.

For convection heat flow from a wall;

$$\varphi = h.A.(T_1 - T)$$

$$\Rightarrow \varphi = \frac{T_1 - T}{R_T} \qquad (1)$$



where: $\varphi \equiv$ rate of heat flow

$h \equiv$ coefficient of heat transfer

$A \equiv$ normal cross section area

$(T_1 - T) \equiv$ temperature gradient

$R_T = \frac{1}{h.A} \equiv$ equivalent thermal resistance thermal capacitance can be expressed as :

$$q = m \cdot C_p \cdot \frac{dT}{dt} \qquad (2)$$

from equations (1) and (2) ;

$$m \cdot C_p \cdot DT = \frac{T_1 - T}{R_T} \qquad \text{OR} \qquad C_T \cdot DT = \frac{T_1 - T}{R_T}$$

$$\therefore T = \frac{T_1}{1 + R_T \cdot C_T \cdot D} \qquad \text{OR} \qquad T = \frac{T_1}{1 + \tau D} \quad (\tau = R_T \cdot C_T)$$

where : $m \equiv$ mass

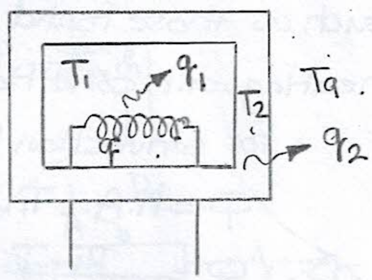
$C_p \equiv$ specific heat at constant pressure

$C_T \equiv$ thermal capacitance ($C_T = m \cdot C_p$).

Example : A heater supplies a heat flux (q) to a room as shown. The temperature of the inside room and the wall is T_1 and T_2, while the ambient temperature T_a. Develop a linear model, considering (q) as input and T_1 as output.

Solution :

For the heat balance of the room ;



$$q = C_{T_1} \cdot DT_1 - q_1$$

OR $$q = C_1 \cdot DT_1 - q_1 \qquad (1)$$

$$q_1 = h_1 \cdot A_1 \cdot (T_1 - T_2) = \frac{T_1 - T_2}{R_1}$$

$$q_2 = h_2 \cdot A_2 \cdot (T_2 - T_a) = \frac{T_2 - T_a}{R_2}$$

and,

for heat balance of the wall ;

$$q_1 = C_2 \cdot DT_2 + q_2 \qquad (2)$$

equations (1) and (2) can be written as :

$$q = C_1 \cdot DT_1 - \left(\frac{T_1 - T_2}{R_1}\right) \qquad * R_1$$

$\Rightarrow R_1 . q = C_1 . R_1 . DT_1 - T_1 + T_2$ (3)

and $q_1 - q_2 = C_2 . DT_2$

$\therefore \Rightarrow \quad C_2 . DT_2 = \dfrac{T_1 - T_2}{R_1} + \dfrac{T_2 - T_a}{R_2}$

$$\left(C_2 . D + \frac{1}{R_1} - \frac{1}{R_2}\right) T_2 = \frac{T_1}{R_1} - \frac{T_a}{R_2} \quad (4)$$

from equations (3) and (4), we get ;

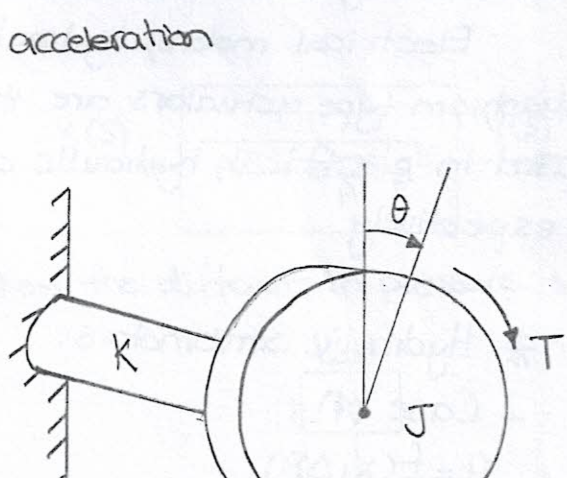
$$q = \frac{\left[(C_1 . R_1 . D - 1) + \dfrac{1}{R_1(C_2 . D + 1/R_1 - 1/R_2)}\right] T_1}{R_1} - \frac{T_a}{R_2(C_2 D + 1/R_1 - 1/R_2)}$$

1.2.5 Angular Displacement.

Σ Torque = Inertia * angular acceleration

$$T - T_s - T_d = J\ddot{\theta}$$

$$T = J\ddot{\theta} + C_t . \dot{\theta} + K_t . \theta \quad (1)$$



where :

$T \equiv$ applied torque (N.m)

$T_s \equiv$ stiffness torque (N.m)

$T_d \equiv$ Damping torque (N.m)

$J \equiv$ Inertia (Kg.m^2)

$\theta \equiv$ angular displacement (rad)

$\dot{\theta} \equiv$ angular velocity (rad/s)

$\ddot{\theta} \equiv$ angular acceleration (rad/s^2)

$C_t \equiv$ torsional damping coefficient

$K_t \equiv$ torsional stiffness coefficient

equation (1) can be written as :

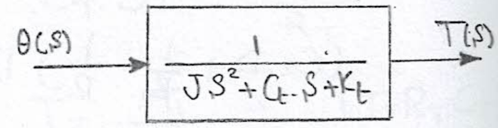
$$T = JD^2 . \theta + C_t . D\theta + K_t . \theta$$

OR

$$\frac{\theta}{T} = \frac{1}{JD^2 + C_t D + K_t}$$

using laplace transform results in;

$$\frac{\theta(s)}{T(s)} = \frac{1}{JS^2 + C_t.S + K_t}$$



1.2.6 Actuators.

An actuator is a control element that used power to drive the system to be controlled. The power requirment may be small as in the case of positioning a control value or larg as in the case where a larg load is to be moved.

Electrical motors, hydraulic servometers and pneumatic diaphram type actuators are the common examples of actuators used in electrical, hydraulic and pneumatic control systems respectively.

* Hydraulic Servomotors.

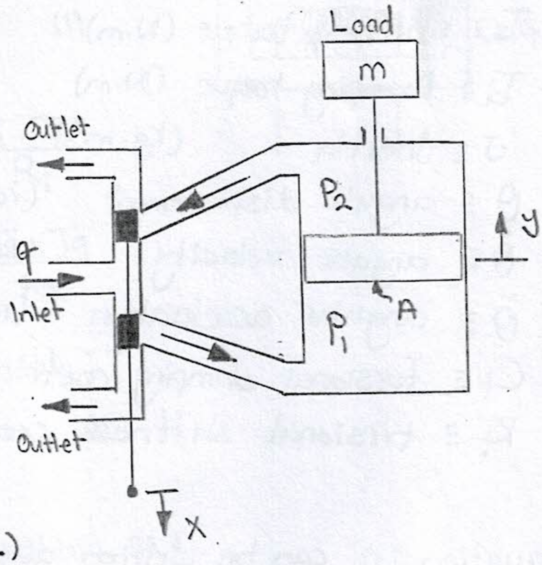
- Case (1):

$q = f(X, \Delta P)$

$\therefore\ q = C_1 X - C_2 \Delta P \quad (1)$

here; $q \equiv$ inlet flow rate

$C_1, C_2 \equiv$ coefficients of charge



the continuity equation is:

$q = A.V = A.\dot{y} = ADy \quad (2)$

the balancing forces :

$$A\Delta P = m\ddot{y} = mD^2y \qquad (3)$$

where : $A \equiv$ cylinder cross section area.

from equations (1), (2) and (3) ;

$$ADy = C_1 X - C_2\left(\frac{mD^2}{A}\right)y$$

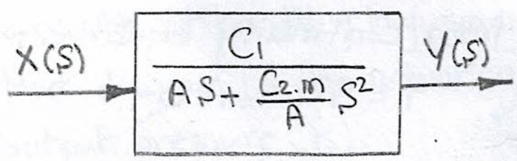
$$\therefore C_1 . X = \left(AD + \frac{C_2 . m}{A} D^2\right)y$$

$\therefore$ System transfer function is :

$$\frac{y}{X} = \frac{C_1}{AD + \frac{C_2 . m}{A} D^2}$$

this transfer function can be written using Laplace transform as :

$$\frac{Y(S)}{X(S)} = \frac{C_1}{AS + \frac{C_2 . m}{A} S^2}$$



where : $S \equiv$ Laplace operator

if there is no load in the system the different in pressure is zero ($\Delta P = 0$).

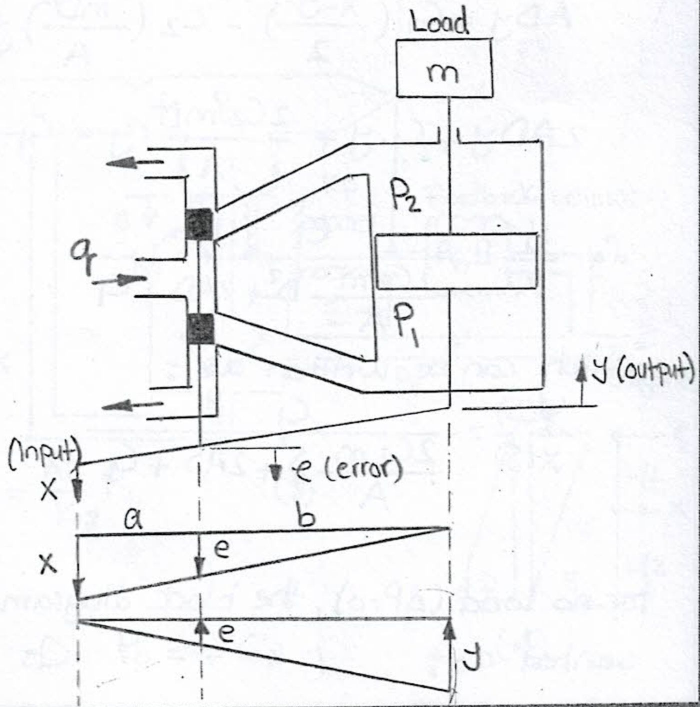
- Case (2) :

$q = f(e, \Delta P)$

$q = C_1 . e \pm C_2 \Delta P \qquad (1)$

To find (e), first we consider no displacement at y ;

$$\Rightarrow \frac{e}{X} = \frac{b}{a+b}$$



$$\therefore \quad e = \frac{b}{a+b}\,x \qquad (a)$$

and then no displacement at x ;

$$\Rightarrow e = \frac{a}{a+b}\,y$$

$$\therefore \quad e = \frac{a}{a+b}\,y \qquad (b)$$

Total (e) can be found from equations (a) and (b) ;

$$e = \frac{b}{a+b}\,x - \frac{a}{a+b}\,y \qquad (2)$$

for special case $a=b \Rightarrow e = \frac{x-y}{2}$

from equation of motion :

$$A\Delta P = m\ddot{y} = mD^2 y \qquad (3)$$

from continuity equation :

$$q = A\dot{y} = ADy \qquad (4)$$

from equations (1) and (4) and when $a=b$ $\left(e = \frac{x-y}{2}\right)$:

$$ADy = C_1\left(\frac{x-y}{2}\right) - C_2\left(\frac{mD^2}{A}\right)y$$

$$2ADy + C_1 . y + \frac{2C_2 . mD^2}{A}\,y = C_1 . x$$

$$\therefore \quad \frac{y}{x} = \frac{C_1}{\frac{2C_2 m}{A}D^2 + 2AD + C_1}$$

which can be written as :

$$\frac{Y(S)}{X(S)} = \frac{C_1}{\frac{2C_2 m}{A}S^2 + 2AS + C_1}$$

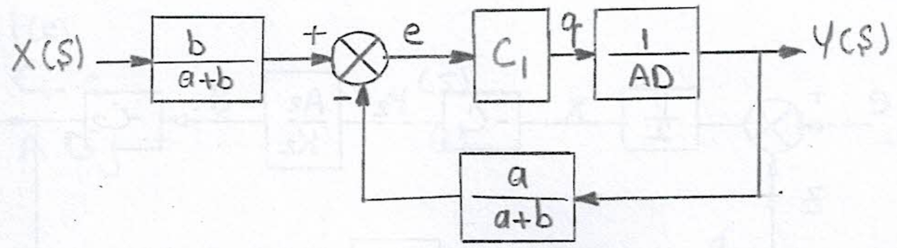
$$X(S) \longrightarrow \boxed{\frac{C_1}{\frac{2C_2 m}{A}S^2 + 2AS + C_1}} \longrightarrow Y(S)$$

for no load $(\Delta P = 0)$, the block diagram of the system can be represented as :

$\therefore\ q = C_1 . e \qquad (1)$

$e = \frac{b}{a+b} x - \frac{a}{a+b} y \qquad (2)$

$q = ADy \qquad (3)$

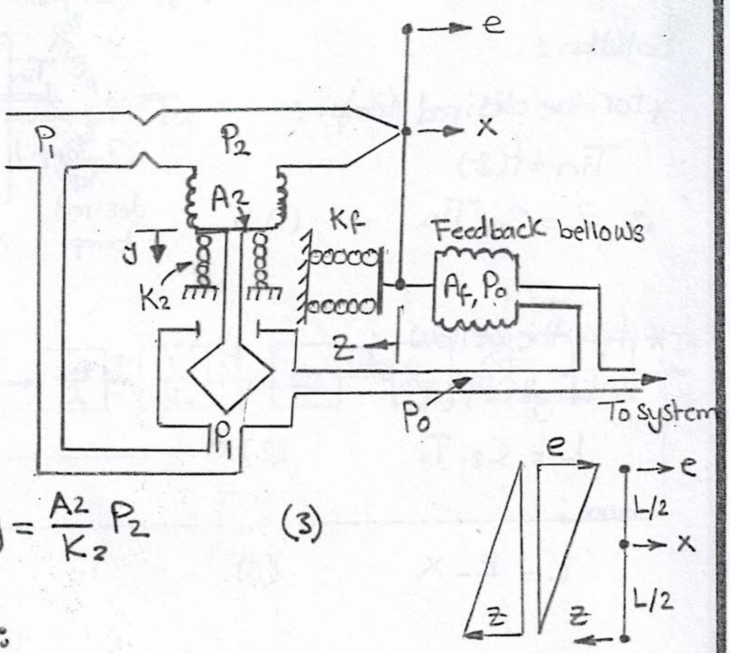


Example: For the control of larg industrial process, where it's necessary to have larg quantities of controlled, so two stages amplifier are used as shown in the figure. The first stage cosists of flapper-type amplifier where the pressure (P_2) controled by the position (x). The second stage is capable of handling larg quantities of flow. Determine the block diagram for the actuating signal (e) as input and the output pressure (P_o).

Solution:

* For the lever with the same length:

$X = \frac{1}{2}(e - z) \qquad (1)$



* For the flapper:

$P_2 = f(x)$ but $P_2 \propto \frac{1}{x}$

$\therefore P_2 = -C_1 X \qquad (2)$

and,

$P_2 . A_2 = K_2 . y \Rightarrow y = \frac{A_2}{K_2} P_2 \qquad (3)$

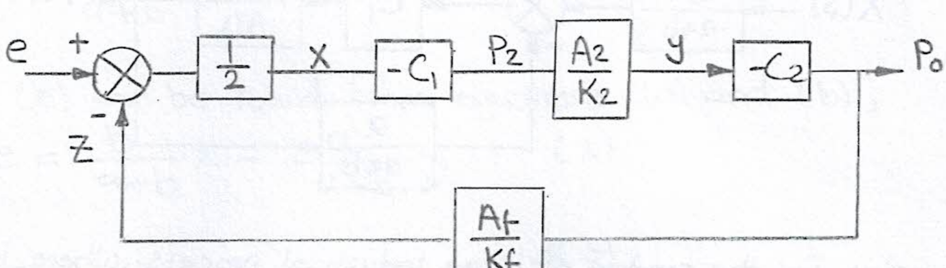
* For the metering valve:

$P_o = f(y)$ but $P_o \propto \frac{1}{y} \Rightarrow P_o = -C_2 . y$

* for the feedback bellow :

$$P_o \cdot A_f = K_f \cdot Z \Rightarrow Z = \frac{A_f}{K_f} P_o \qquad (5)$$

thus, for these equations we can represent the following block diagram :



Example : The system shown in the figure controlling the output temperature (T_o) of a chamber, such as an industrial oven. The desired temperature (T_{in}) is indicated by the pointer on the control arm. The bellow is filled with a liquid expand as (T_o) increase. Obtain the block diagram for reference temperature (T_{in}) to the controlled temperature (T_o).

Solution :

* for the desired temp. :

$T_{in} = f(Z)$

$\therefore Z = C_1 \cdot T_{in} \qquad (1)$

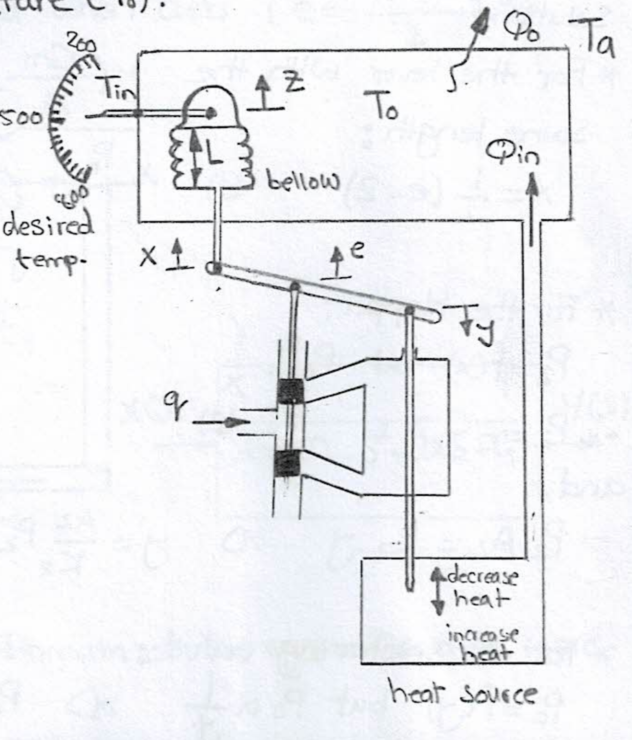
* for the bellow :

$\text{length} = f(T_o)$

$L = C_2 \cdot T_o \qquad (2)$

and ;

$L = Z - X \qquad (3)$



* for the rate of heat flow in chamber :

$\varphi_{in} = f(y)$

$\varphi_{in} = C_3 . y \qquad (4)$

* for the actuator :

$q = f(e)$

$q = C_4 \, e \qquad (5)$

$q = A . D y \qquad (6)$

* for the lever with $(a=b)$: $\because e = \frac{b}{a+b} x - \frac{a}{a+b} y$

$\therefore e = \frac{x-y}{2} \qquad (7)$

* for the chamber heat balance :

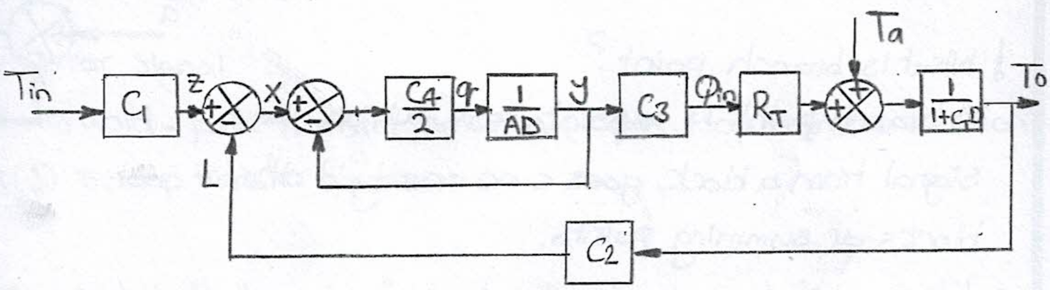
$\varphi_{in} = \varphi_o + C_T . D T_o$

$\because \varphi_o = \frac{T_o - T_a}{R_T}$

$\therefore R_T . \varphi_{in} + T_a = (1 + C_T . D) T_o$

OR

$$T_o = \frac{R_T}{1+C_T D} \varphi_{in} + \frac{1}{1+C_T D} T_a \qquad (8)$$

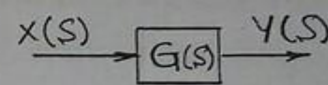


2.3 Block Diagram Representation.

It is important to note that blocks can be connected in series only if the output of one block is not affected by the next following block. If there are any loading effects between the components, it is necessary to combine these components into a single block. A complicated block diagram involving many feedback loops can be simplified by a step-by-step rearrangment, using rules of block diagram algebra. Some of these important rules are given below:

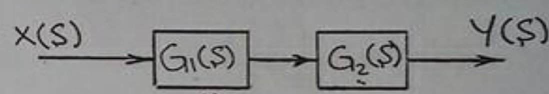
1. Input - output relation

$$Y(S) = G(S).X(S)$$



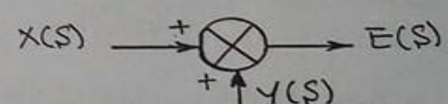
2. Multiplication

$$Y(S) = [G_1(S).G_2(S)]\,X(S)$$

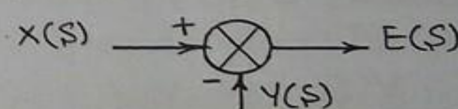


3. Addition and Substraction.

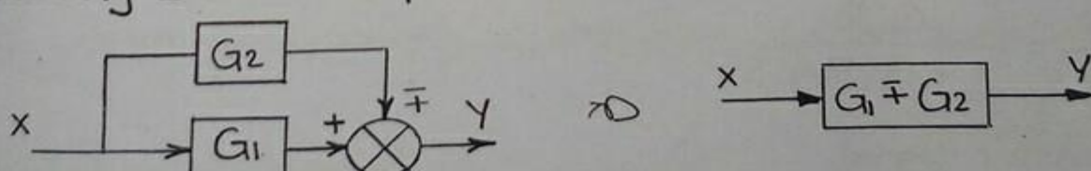
$$E(S) = X(S) + Y(S)$$



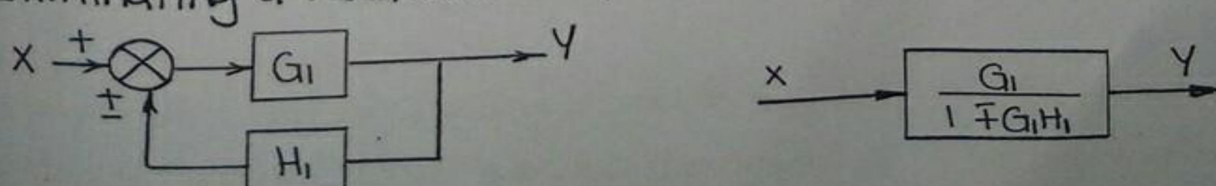
$$E(S) = X(S) - Y(S)$$



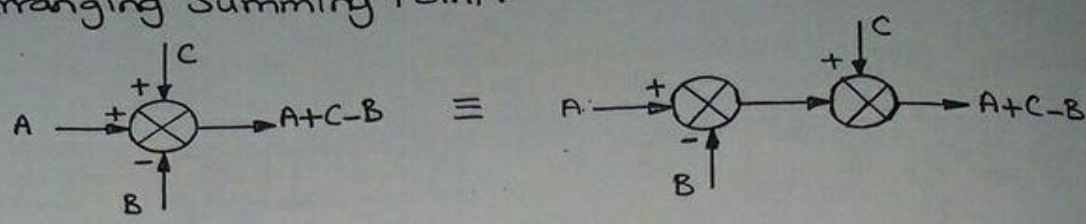
4. Combining Blocks in parallel (Forward Loop).

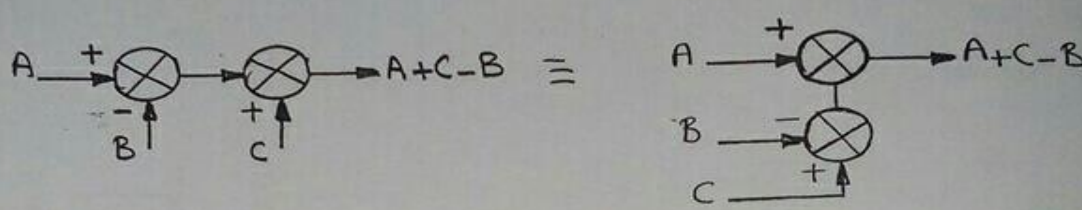


5. Eliminating a Feedback Loop.

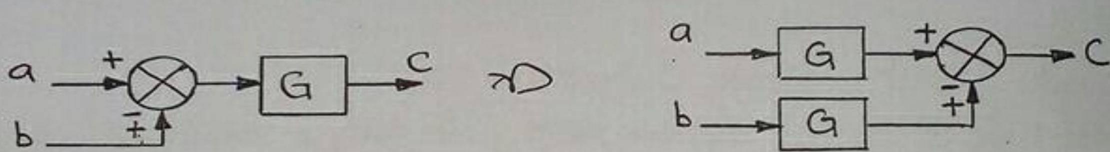


6. Rearranging Summing Point.

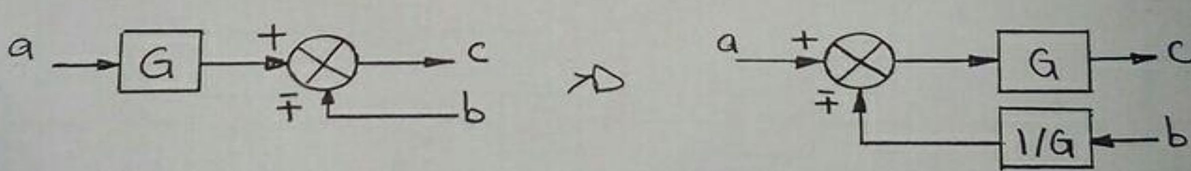




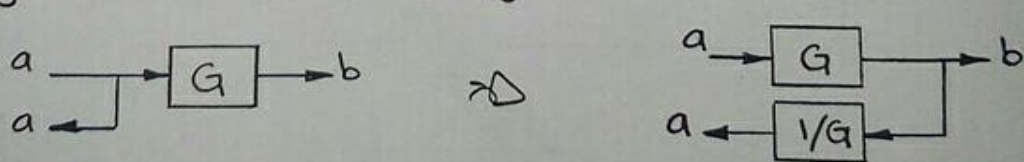
7. Moving a Summing Point Beyond a Block.



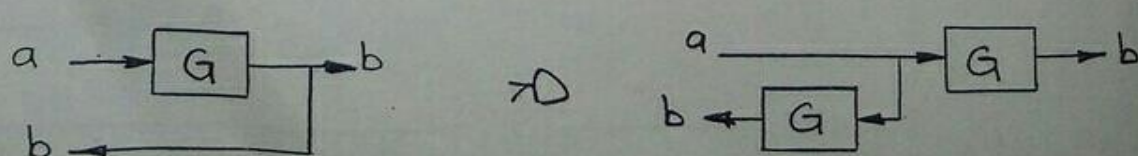
8. Moving a Summing Point Ahead of a Block.



9. Moving a Pick-off Point Beyond a Block.

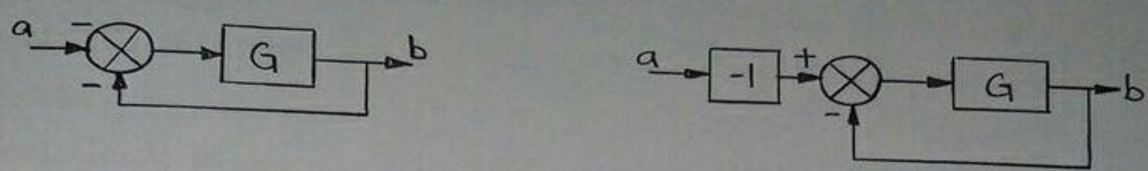


10. Moving a Take-off Point Ahead of a Block.

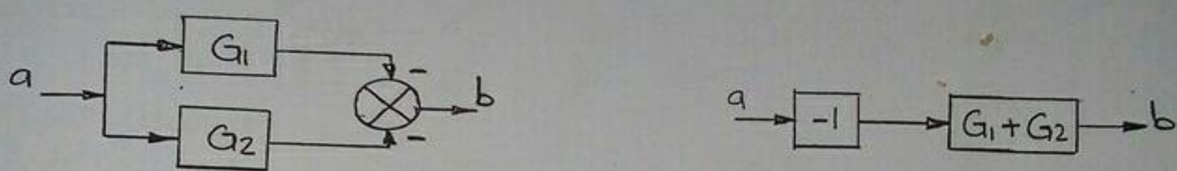


11. Special Cases.

a) Input signal must be positive:



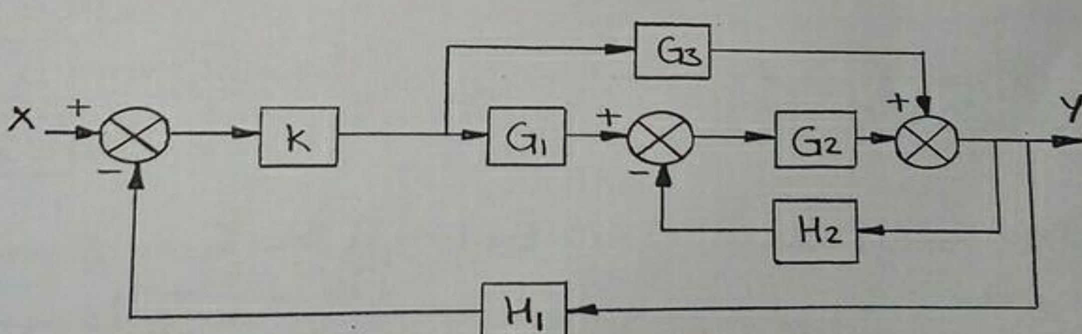
b) Forward_Loop



2.4 Single Input - Single Output System.

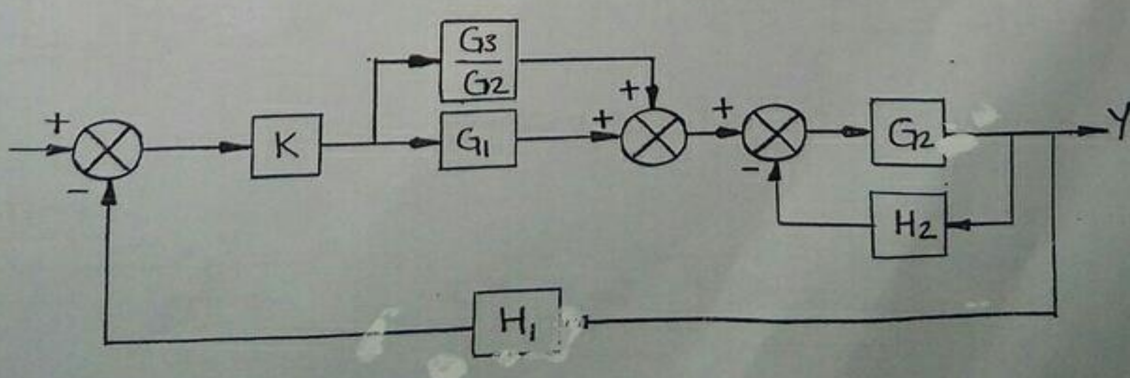
This type of control system have just single input and single output.

Example: Find the transfer function of the block diagram by reducing it.

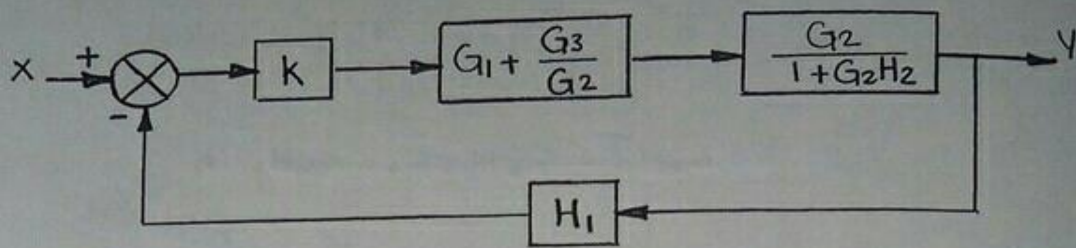


Solution:

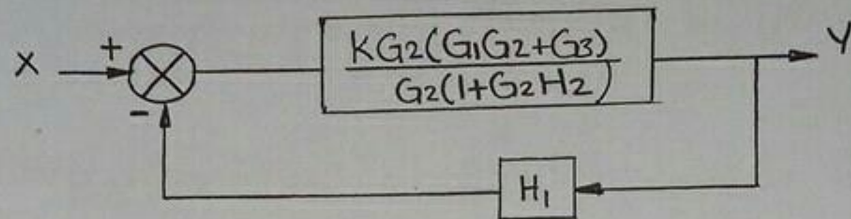
Step (1) :-



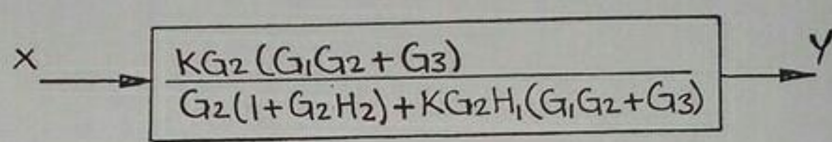
Step(2):



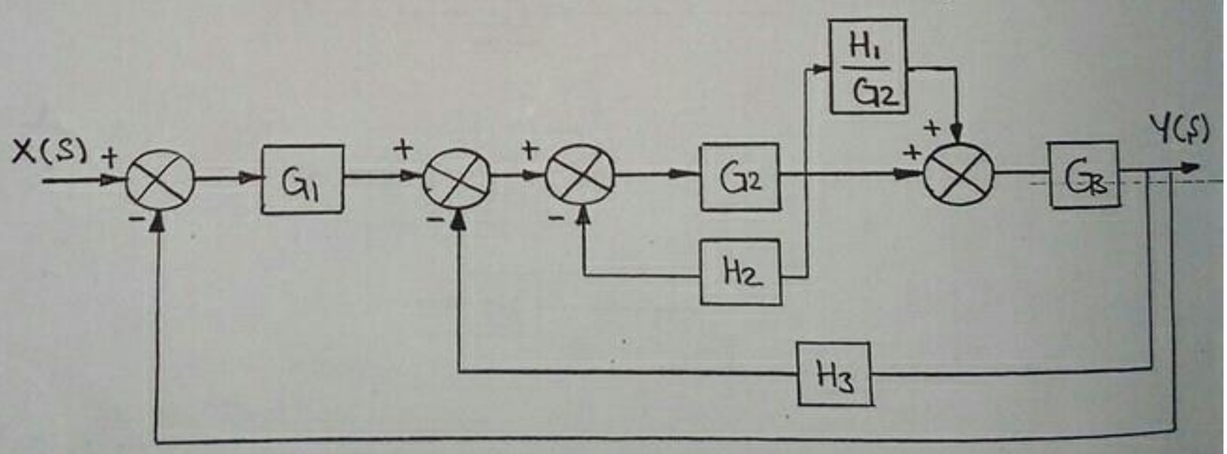
Step(3):



Step(4):

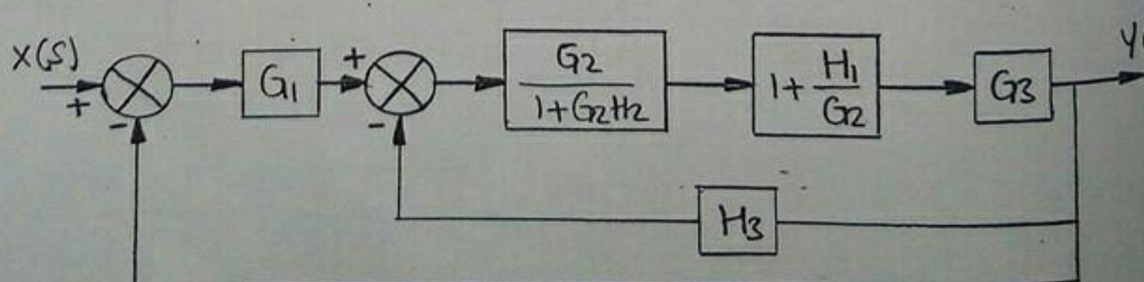


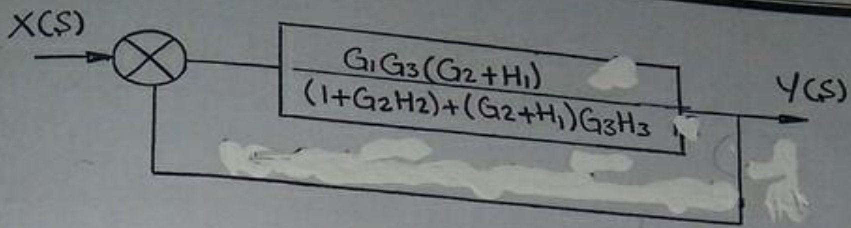
Example: Find the transfer function of the block diagram shown below.



Solution:

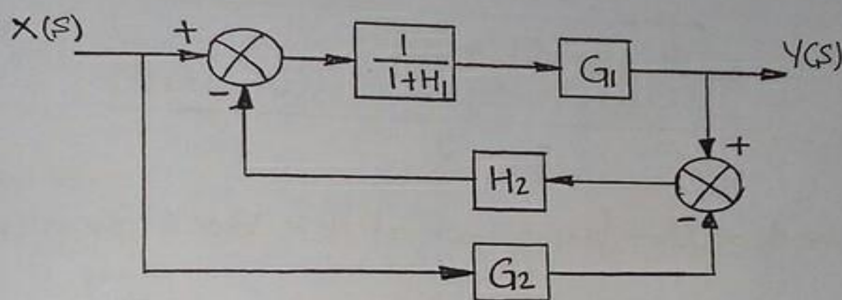
Step(1)

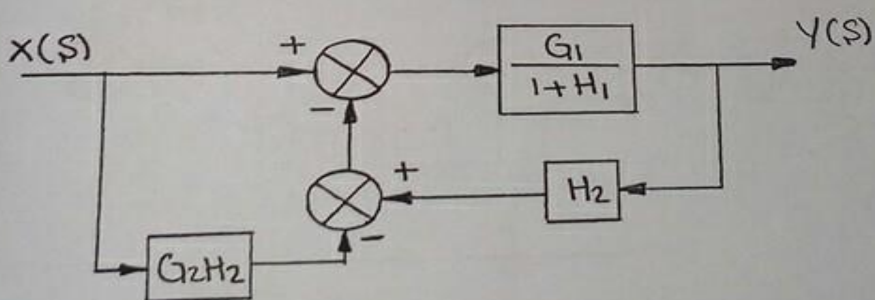


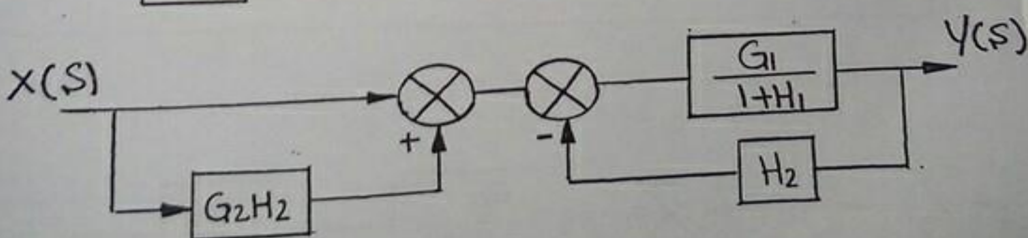


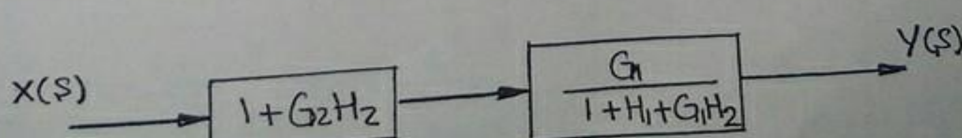
$$\frac{Y(S)}{X(S)} = \frac{G_1G_2G_3 + G_1G_3H_1}{1 + G_2H_2 + G_2G_3H_3 + G_3H_1H_3 + G_1G_2G_3 + G_1G_3H_1}$$

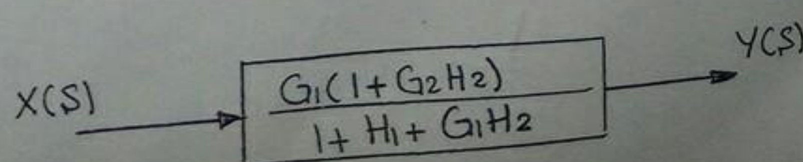
Example: Find the transfer function of the following block diagram.



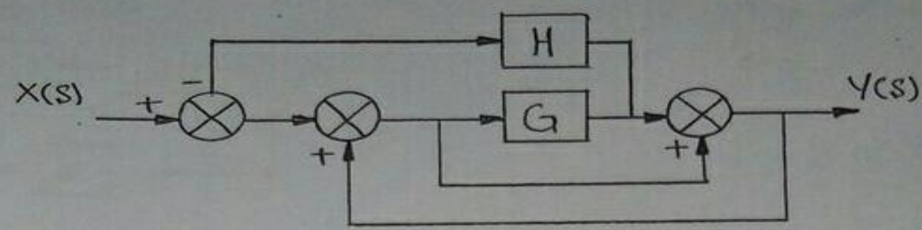


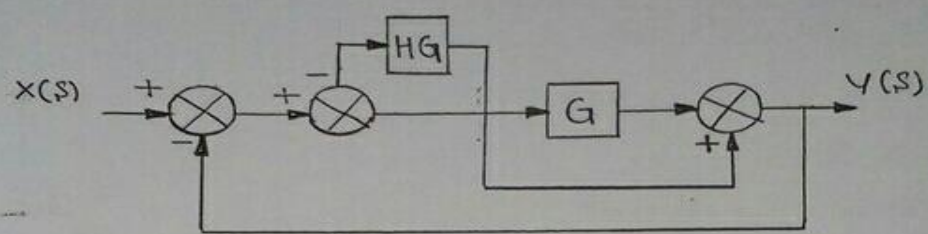


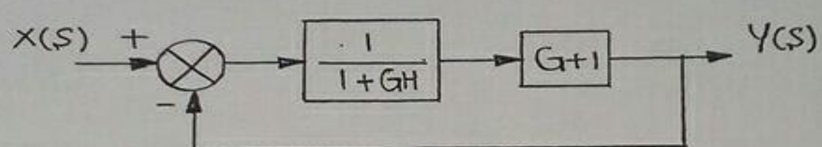


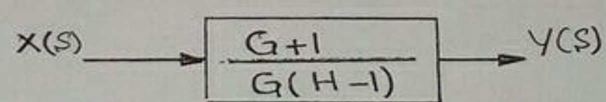


Example : Find the transfer function.



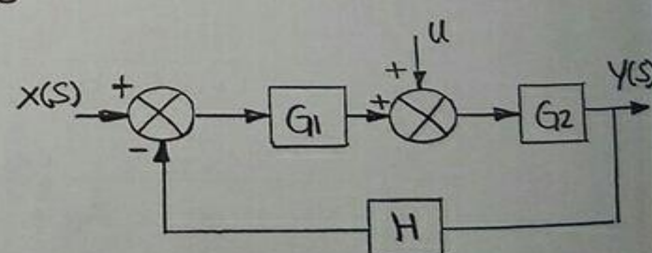






2.5 Two Input - One output Systems

For the system shown here, there are two inputs $X(S)$ and $U(S)$ with one output $Y(S)$. So,



$$Y(S) = C.X(S) + C.U(S)$$

where:

$Y(S) \equiv$ output signal

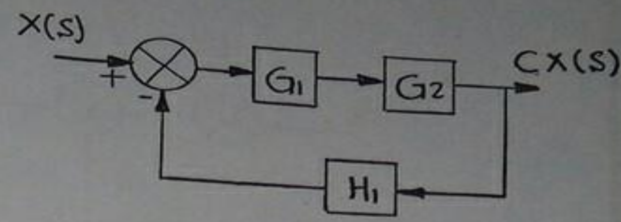
$X(S), U(S) \equiv$ input signal

$C\,X(S) \equiv$ T.F. of the system when $U(S)=0$

$C\,U(S) \equiv$ T.F. of the system when $X(S)=0$

a) when $U(S)=0$

The transfer function of such this loop is ;

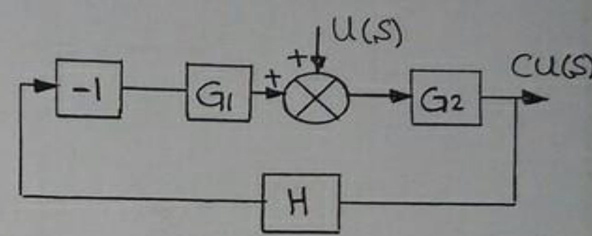


$$\frac{CX(S)}{X(S)} = \frac{G_1G_2}{1+G_1G_2H}$$

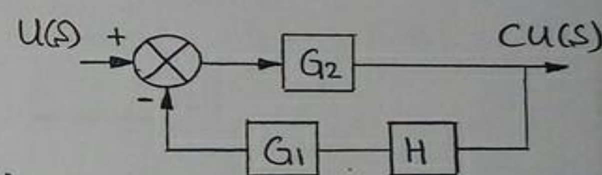
$$\therefore CX(S) = \frac{G_1G_2}{1+G_1G_2H} \cdot X(S)$$

b) when $X(S)=0$

The transfer function of this loop is ;



$$\frac{CU(S)}{U(S)} = \frac{G_2}{1+G_1G_2H}$$



$$\therefore CU(S) = \frac{G_2}{1+G_1G_2H} \cdot U(S)$$

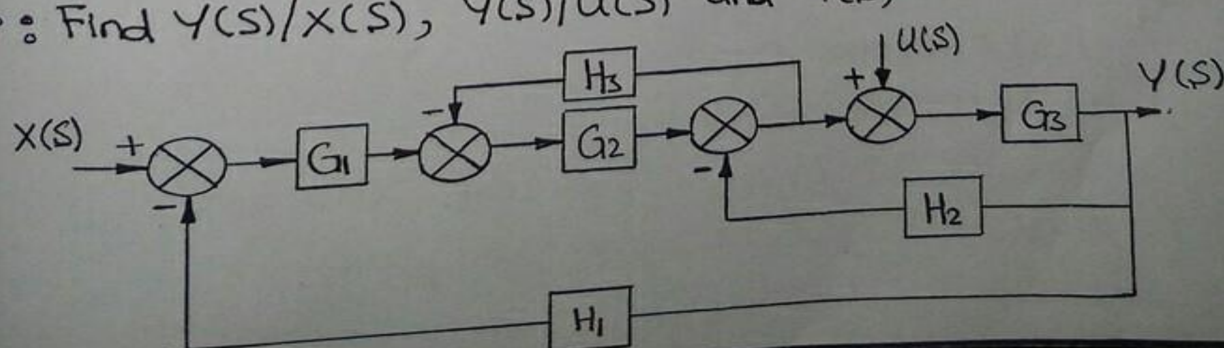
now, the overall transfer function is,

$$Y(S) = CX(S) + CU(S)$$

$$= \frac{G_1G_2}{1+G_1G_2H} \cdot X(S) + \frac{G_2}{1+G_1G_2H} U(S)$$

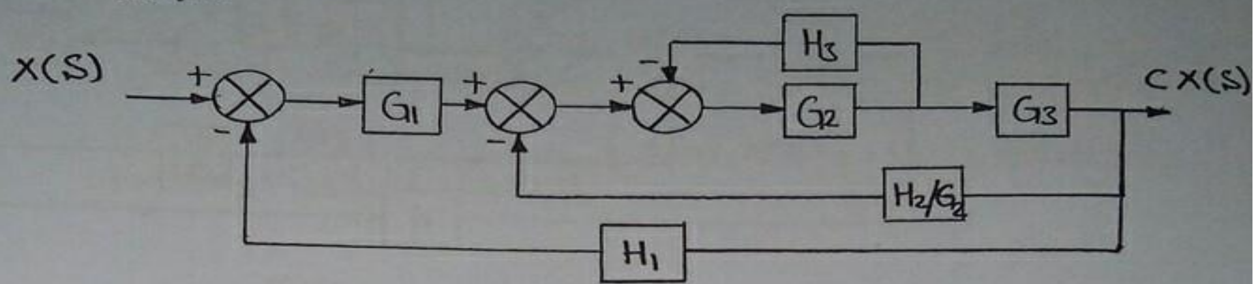
$$= \frac{G_2}{1+G_1G_2H}\left[G_1X(S)+U(S)\right]$$

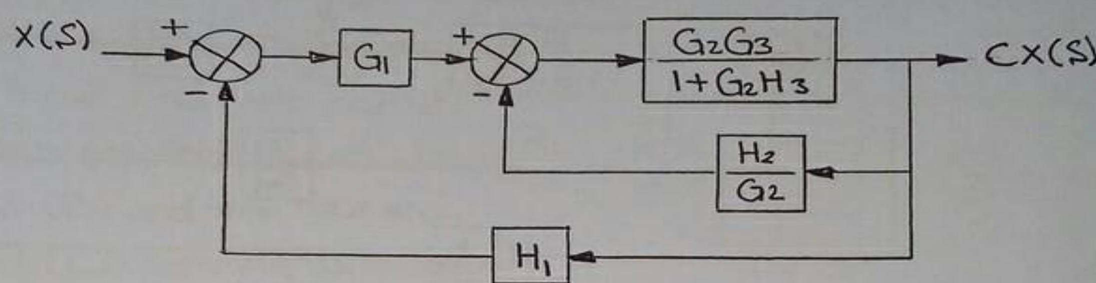
Example : Find $Y(S)/X(S)$, $Y(S)/U(S)$ and $Y(S)$

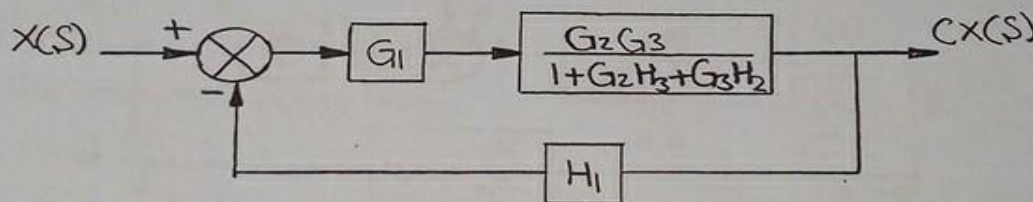


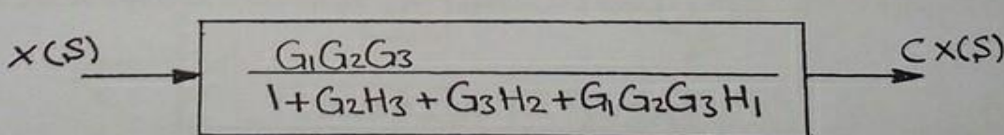
Solution:

I) when $U(S)=0$





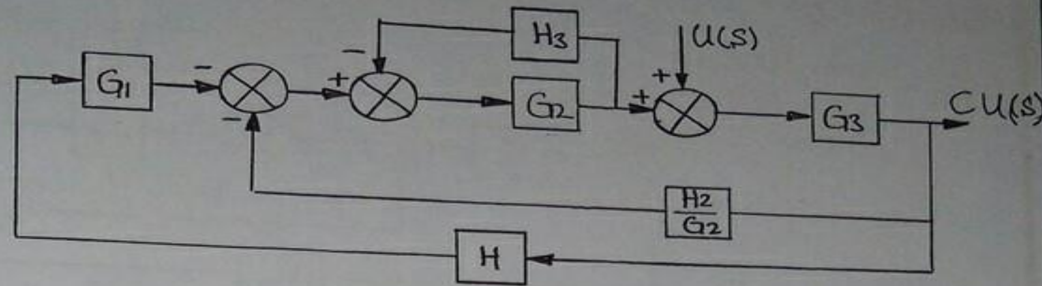


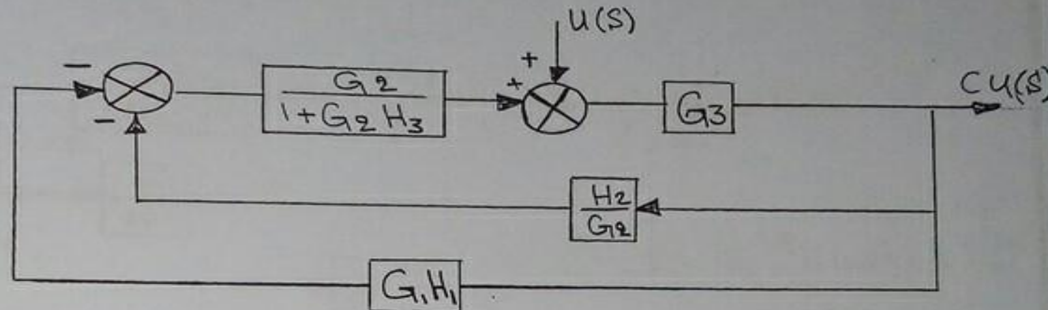


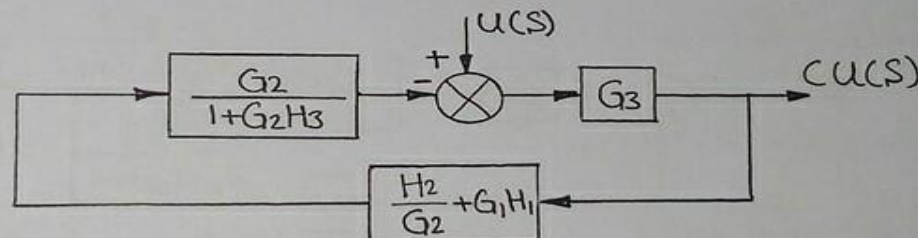
$$\therefore \frac{CX(S)}{X(S)} = \frac{Y(S)}{X(S)} = \frac{G_1G_2G_3}{1+G_2H_3+G_3H_2+G_1G_2G_3H_1}$$

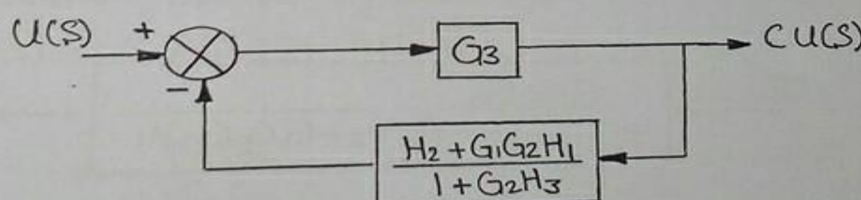
$$\therefore CX(S) = \frac{G_1G_2G_3}{1+G_2H_3+G_3H_2+G_1G_2G_3H_1} \cdot X(S)$$

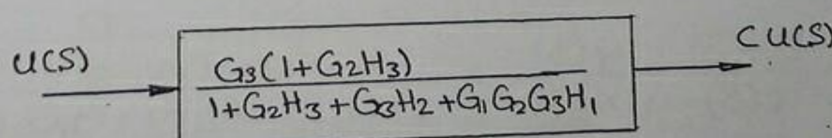
II) when $X(S)=0$











$$\therefore \frac{C_U(S)}{U(S)} = \frac{C(S)}{U(S)} = \frac{G_3(1+G_2H_3)}{1+G_2H_3+G_3H_2+G_1G_2G_3H_1}$$

$$\Rightarrow C_U(S) = \frac{G_3(1+G_2H_3)}{1+G_2H_3+G_3H_2+G_1G_2G_3H_1} \cdot U(S)$$

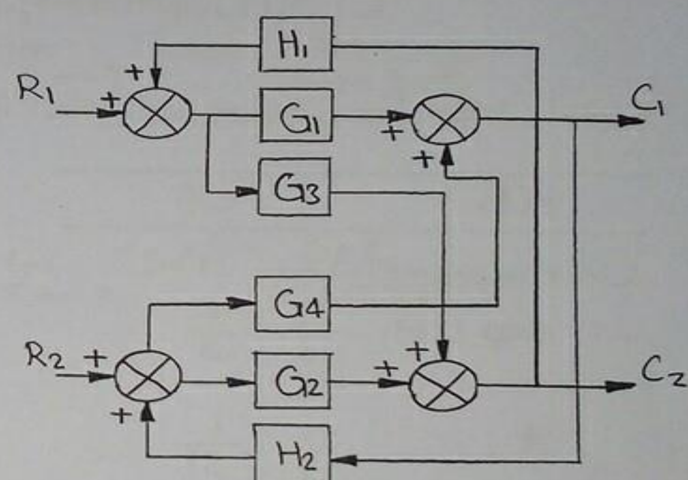
$$\therefore \quad Y(S) = C\,X(S) + C\,U(S)$$

$$= \frac{G_1G_2G_3}{1+G_2H_3+G_3H_2+G_1G_2G_3H_1}\,X(S) + \frac{G_3(1+G_2H_3)}{1+G_2H_3+G_3H_2+G_1G_2G_3H_1}\,U(S)$$

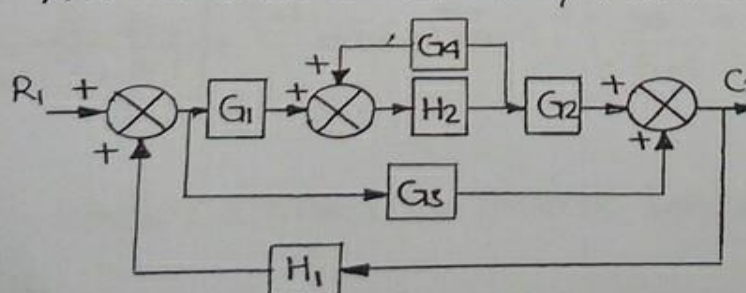
$$= \frac{G_3}{1+G_2H_3+G_3H_2+G_1G_2G_3H_1}\cdot\left[G_1G_2\cdot X(S) + (1+G_2H_3)\,U(S)\right]$$

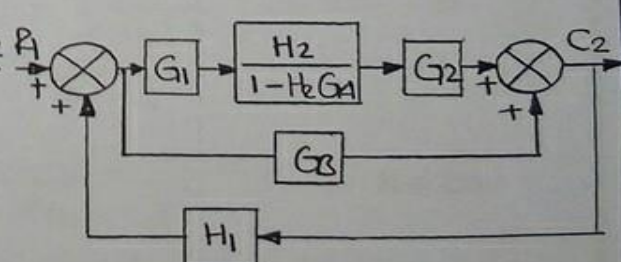
2.6 Two Input - Two output Systems.

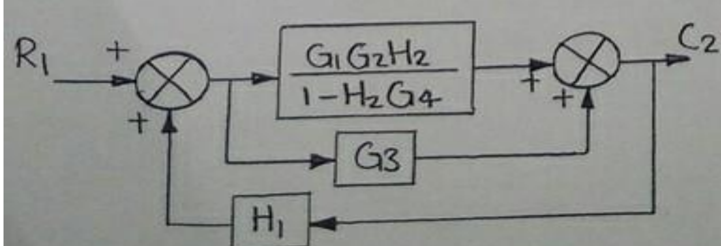
For the system shown, there are two input and two output. So, we assume that ($R_2=0$) ; i.e. one input (R_1) and then take each of (C_1) and (C_2) seperatly with (R_1) to evaluate $\frac{C_1}{R_1}$ and $\frac{C_2}{R_1}$. Thereafter, we assume ($R_1=0$) to find $\frac{C_1}{R_2}$ and $\frac{C_2}{R_2}$. For the example given here, we can find $\frac{C_2}{R_1}$:

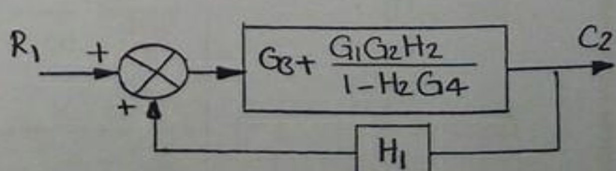


* assume $R_2=0$ and $C_1=0$, then re-arrang the system to get ;

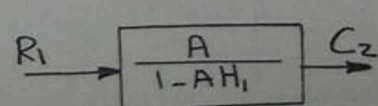








let $A = G_3 + \frac{G_1G_2H_2}{1-H_2G_4}$



2.7 Laplace Transformation

The Laplace Transformation can be used for the solution of linear differential equations. We are concerned here with the transformation of function of time and their time derivatives into functions of a complex variable (S). The solution as a function of time is then obtained by taking the inverse Laplace transformation. The Laplace transform of f(t) is given by :

$$L[f(t)] = F(S) = \int_0^{\infty} f(t).e^{-St}.dt$$

Laplace transform pairs are given in the table below.

f(t)	F(S)
Unit impulse $\delta(t)$	1
Unit step $1(t)$	$\frac{1}{S}$
t	$\frac{1}{S^2}$
t^n	$\frac{n!}{S^{n+1}}$
e^{at}	$\frac{1}{S-a}$
$\cos \omega t$	$\frac{S}{S^2+\omega^2}$
$\sin \omega t$	$\frac{\omega}{S^2+\omega^2}$
$\cosh at$	$\frac{S}{S^2-a^2}$
$\sinh at$	$\frac{a}{S^2-a^2}$

f(t)	F(S)
$t.e^{at}$	$\frac{1}{(S-a)^2}$
$t^n.e^{at}$	$\frac{n!}{(S-a)^{n+1}}$
$e^{at}.\sin \omega t$	$\frac{\omega}{(S-a)^2+\omega^2}$
$e^{at}.\cos \omega t$	$\frac{S-a}{(S-a)^2+\omega^2}$

Example: Find the inverse Laplace transform of

$$F(S) = \frac{S+3}{(S+1)(S+2)}$$

Solution: The partial-fraction expansion of F(S) is

$$F(S) = \frac{S+3}{(S+1)(S+2)} = \frac{K_1}{S+1} + \frac{K_2}{S+2}$$

the constant K_1 and K_2 can be found by;

$$* \; K_1 = \lim_{S \to -1} (S+1) \cdot F(S) = \lim_{S \to -1} \cancel{(S+1)} \cdot \frac{S+3}{(S+2)\cancel{(S+1)}}$$

$$\Rightarrow K_1 = \lim_{S \to -1} \frac{S+3}{S+2} = 2$$

$$* \; K_2 = \lim_{S \to -2} (S+2) \cdot F(S) = \lim_{S \to -2} \cancel{(S+2)} \frac{S+3}{(S+1)\cancel{(S+2)}}$$

$$\Rightarrow K_2 = \lim_{S \to -2} \frac{S+3}{S+1} = -1$$

thus; $F(S) = \frac{2}{S+1} - \frac{1}{S+2}$

$$\Rightarrow \therefore f(t) = L^{-1}[F(S)] = L^{-1}\left[\frac{2}{S+1}\right] + L^{-1}\left[\frac{-1}{S+2}\right] = 2e^{-t} - e^{-2t}$$

Example: Obtain the inverse Laplace transform of:

$$F(S) = \frac{1}{(S^2+6S+8)(S+6)}$$

Solution: We can Write;

$$F(S) = \frac{1}{(S+2)(S+4)(S+6)}$$

now, the partial fraction equation is;

$$F(S) = \frac{K_1}{S+2} + \frac{K_2}{S+4} + \frac{K_3}{S+6}$$

and the constant can be found;

$$K_1 = \lim_{S \to -2} (S+2) \cdot F(S) = \lim_{S \to -2} (\cancel{S+2}) \cdot \frac{1}{(\cancel{S+2})(S+4)(S+6)}$$

$$= \lim_{S \to -2} \cdot \frac{1}{(S+4)(S+6)} = \frac{1}{8}$$

$$K_2 = \lim_{S \to -4} (S+4) \cdot F(S) = \lim_{S \to -4} (\cancel{S+4}) \cdot \frac{1}{(S+2)(\cancel{S+4})(S+6)}$$

$$= \lim_{S \to -4} \cdot \frac{1}{(S+2)(S+6)} = -\frac{1}{4}$$

$$K_3 = \lim_{S \to -6} (\cancel{S+6}) \cdot \frac{1}{(S+2)(S+4)(\cancel{S+6})} = \lim_{S \to -6} \cdot \frac{1}{(S+2)(S+4)} = \frac{1}{8}$$

$$\therefore F(S) = \frac{1/8}{S+2} - \frac{1/4}{S+4} + \frac{1/8}{S+6}$$

$\therefore$ the Laplace transform is :

$$f(t) = \frac{1}{8} e^{-2t} - \frac{1}{4} e^{-4t} + \frac{1}{8} e^{-6t}$$

Example: Find the inverse Laplace transformation of the equation;

$$F(S) = \frac{10}{(S+2)(S+1)^3}$$

Solution: For the repeated zeroes, the correspoding partial fraction expansion is;

$$F(S) = \frac{C_n}{(S-r)^n} + \frac{C_{n-1}}{(S-r)^{n-1}} + \cdots + \frac{C_1}{S-r} + \frac{K_1}{S-r_1} + \cdots$$

$$C_n = \lim_{S \to r} \left[(S-r)^n \cdot F(S) \right]$$

$$C_{n-1} = \lim_{S \to r} \left[\frac{d}{dS} \left((S-r)^n . F(S) \right) \right]$$

$$C_{n-k} = \lim_{S \to r} \left[\frac{1}{K_i} \cdot \frac{d^k}{dS^k} \left((S-r)^n . F(S) \right) \right]$$

So, for this example ;

$$F(S) = \frac{C_3}{(S+1)^3} + \frac{C_2}{(S+1)^2} + \frac{C_1}{(S+1)} + \frac{K_1}{S+2}$$

$$\therefore C_3 = \lim_{S \to -1} (S+1)^3 . \frac{10}{(S+2)(S+1)^3} = \lim_{S \to -1} \frac{10}{S+2} = 10$$

$$C_2 = \lim_{S \to -1} \frac{d}{dS} \left[(S+1)^3 . F(S) \right] = \lim_{S \to -1} \frac{d}{dS} \left[\frac{10}{S+2} \right] .$$

$$= \lim_{S \to -1} . \frac{-10}{(S+2)^2} = -10$$

$$C_1 = \lim_{S \to -1} \left[\frac{1}{2!} \cdot \frac{d^2}{dS^2} \left((S+1)^3 . F(S) \right) = \frac{1}{2} \lim_{S \to -1} . \frac{d^2}{dS^2} \left(\frac{10}{S+2} \right)$$

$$= \frac{1}{2} \lim_{S \to -1} \left(\frac{10 \times 2 \times (S+2)}{(S+2)^4} \right) = 10$$

$$K_1 = \lim_{S \to -2} (S+2) . F(S) = \lim_{S \to -2} . \frac{10}{(S+1)^3} = -10$$

$$\therefore F(S) = \frac{-10}{S+2} + \frac{10}{(S+1)^3} - \frac{10}{(S+1)^2} + \frac{10}{S+1}$$

$$\therefore f(t) = -10 e^{-2t} + 10 \frac{t^2}{2} e^{-t} - 10 t e^{-t} + 10 e^{-t}$$

Example : Find $f(t)$ for the function $F(S) = \dfrac{20}{(S^2+4S+13)(S+6)}$

Solution : For complex conjugate zero, the partial fraction expansion is;

$$F(S) = \frac{K_c}{S-a-jb} + \frac{K_{-c}}{S-a+jb}$$

therefore,

$$F(S) = \frac{20}{(S+2-3j)(S+2+3j)(S+6)} = \frac{K_c}{S+2-3j} + \frac{K_{-c}}{S+2+3j} + \frac{K_1}{S+6}$$

$$K_c = \frac{1}{2bj} |K(a+bj)| e^{\alpha j}$$

$$K_{-c} = -\frac{1}{2bj} |K(a+bj)| e^{-\alpha j}$$

or $$L^{-1}\left[\frac{K_c}{S+a+bj} + \frac{K_{-c}}{S+a-bj}\right] = \frac{1}{b} |K(a+bj)| e^{at} . \sin(bt+\alpha)$$

where;

$$K(a+bj) = \lim_{S \to -2+3j} (S^2+4S+13) . F(S)$$

$$= \lim_{S \to -2+3j} . \frac{20}{S+6} = \frac{20}{4+3j} * \frac{4-3j}{4-3j}$$

$\therefore K(a+bj) = 3.2 - 2.4j$ $\therefore |K(a+bj)| = \sqrt{3.2^2 + 2.4^2} = 4$

$$\alpha = \tan^{-1} \frac{-2.4}{3.2} = -36.86^\circ$$

$$\therefore L^{-1}\left[\frac{K_c}{S+2-3j} + \frac{K_{-c}}{S+2+3j}\right] = \frac{1}{3} * 4 * e^{-2t} . \sin(3t - 36.86)$$

$$K_1 = \lim_{S \to -6} (S+6) . F(S) = \frac{20}{S^2+4S+13} = 0.8$$

$$\therefore f(t) = 0.8 e^{-6t} + \frac{4}{3} e^{-2t} . \sin(3t - 36.86)$$

3.1 Transient Response of Control Systems.

The output variation during the time, it takes to achieve its final value, is called as transient response. The time required to achieve the final value is called "transient period". This can also be defined as that part of the time response which decays to zero after same time as system output reaches to its final value.

Successfulness and accuracy of system depends on the final value reached by the system output which should be very close to what is desired from that system. While reaching to its final value, in the mean time, output should behave smoothly. Thus, final state achieved by the output is called "steady state" while output variations within the time it takes to achieve the steady state is called "transient response" of the system.

3.2 Steady State Response of Control Systems.

It is that part of the time response which remains after complete "transient response" vanishes from the system output. This also can be defined as response of the system as time approaches infinity from the time at which transient response completely dies out. The steady state response is generally the final value achieved by the system output.

Hence, total time response $Y(t)$ We can write as,

$$Y(t) = Y_{ss} \text{ (steady state response)} + Y_t(t) \text{ (transient response)}$$

The difference between the desired output and actual output of the system is called "steady state error" which denoted as E_{ss}. This error indicates the accuracy and plays an important role is designing the system.

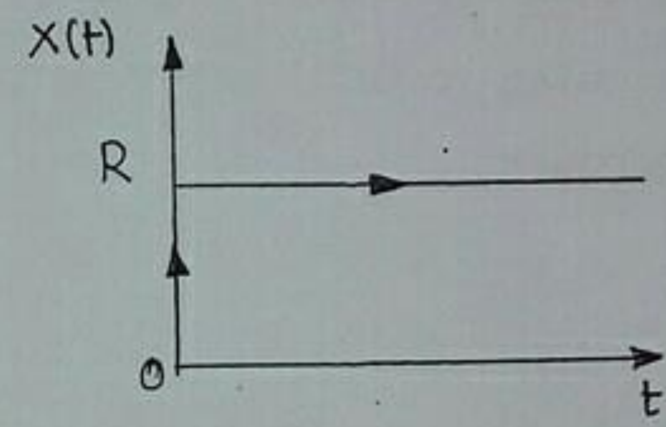
3.3 Standard Test Inputs.

In practice, many signals are available which are the functions of time and can be used as reference inputs for the various control systems. The evaluation of the system can be done on the basis of the response given by the system to the standard test inputs.

3.3.1 Step Input Signal :

The step input signal represents an instantenous charge in the reference input variable. For example, if the input is an angular position of a mechanical shaft, the step input represents the sudden rotation of the shaft. The mathmatical representation of a step function is;

$$X(t) = \begin{cases} R & t \geq 0 \\ 0 & t < 0 \end{cases}$$



where, "R" is a constant.

∴ the Laplace tranform of a constant value;

$$X(S) = \frac{R}{S}$$

and for a unit step signal $X(t) = 1$

$$\Rightarrow \quad X(S) = \frac{1}{S}$$

∴ the transfer function of the system in S-domain :

$$\therefore \frac{Y(S)}{X(S)} = G(S)$$

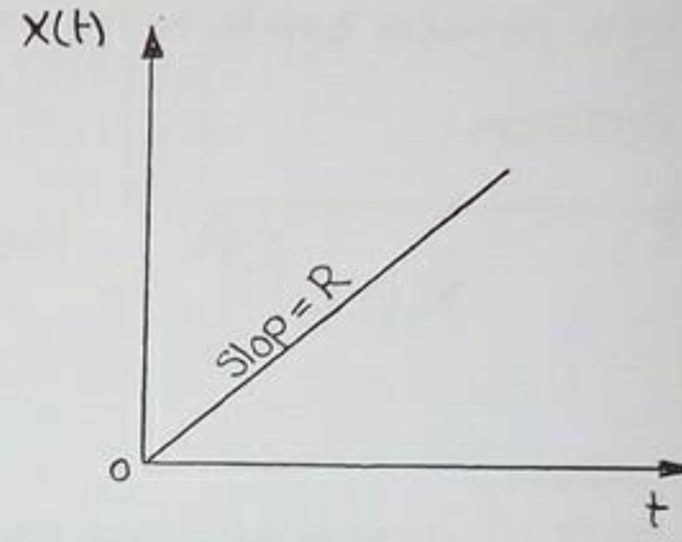
$$\Rightarrow Y(S) = G(S) \cdot X(S) = G(S) \cdot \frac{1}{S}$$

$$\therefore y(t) = L^{-1}\left(\frac{G(S)}{S}\right).$$

3.3.2 Ramp Input Signal:

In the case of ramp signal, the signal is considered to have a constant change in value with respect to time. mathmatically, a ramp function is represented by :

$$X(t) = \begin{cases} R.t & t \geq 0 \\ 0 & t < 0 \end{cases}$$



where, "R" is a constant.

For unit ramp Input; $X(t) = t$

$\therefore$ the Laplace transform for such this function is :

$$X(S) = \frac{1}{S^2}$$

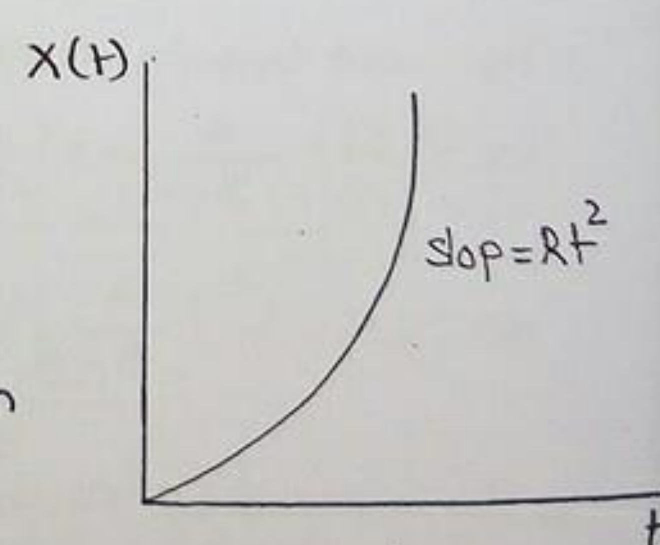
$\Rightarrow \therefore Y(S) = G(S).X(S) \quad \Rightarrow \therefore Y(S) = \frac{G(S)}{S^2}$

$$\therefore y(t) = L^{-1}\left(\frac{G(S)}{S^2}\right)$$

3.3.3 Parabolic Input Signal:

The mathmatical representation of a parabolic input function is :

$$X(t) = \begin{cases} R.t^2 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



For a unit parabolic input :

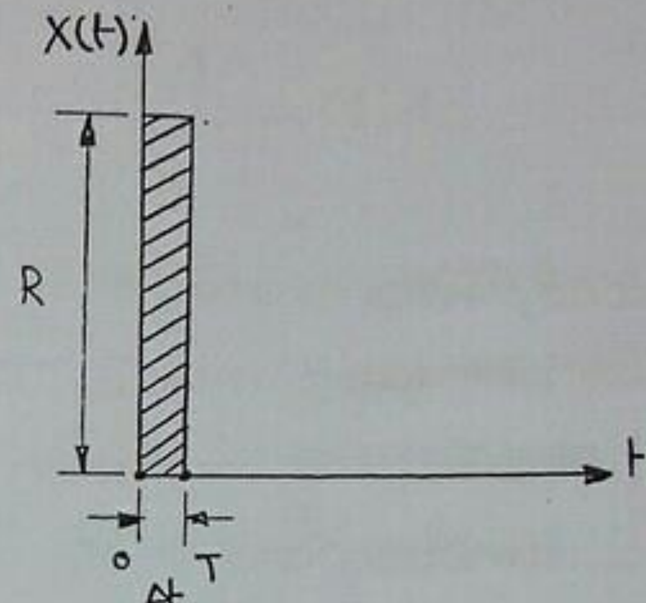
$X(t) = t^2 \Rightarrow$ the Laplace transform is :

$$X(S) = \frac{2}{S^3} \Rightarrow Y(S) = \frac{2\,G(S)}{S^3}$$

$$\Rightarrow \therefore y(t) = L^{-1}\left(\frac{2.G(S)}{S^3}\right)$$

3.3.4 Impulse Input Signal :

There is a jump at the time of application of impulse and the $X(t)$. It is the input applied instantaneously (for short duration of time) of very high amplitude as show in the figure. However, a stable system will return again to its equilibrium position:



$$X(t) = \begin{cases} R & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases}$$

where;

R : pulse height

T : Small duration time

for a unit Impulse signal; $X(S) = 1$

$$Y(S) = G(S) \cdot X(S) = G(S)$$

$$\therefore y(t) = L^{-1}(G(S))$$

- Respose of First-Order System.

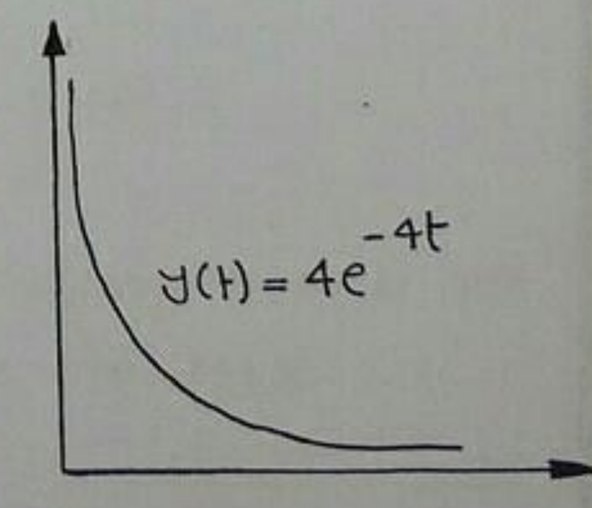
Example: For the system with transfer function $G(S) = \frac{4}{S+4}$, find it's unit impulse response and unit step response.

Solution:

* For unit Impulse response; $X(S) = 1$

$$\therefore Y(S) = \frac{4}{S+4} * 1$$

$$y(t) = L^{-1}\left(\frac{4}{S+4}\right) = 4e^{-4t}$$



* For unit step response; $X(S) = \frac{1}{S}$

$$Y(S) = G(S) \cdot \frac{1}{S} = \frac{4}{S(S+4)} = \frac{K_1}{S} + \frac{K_2}{S+4}$$

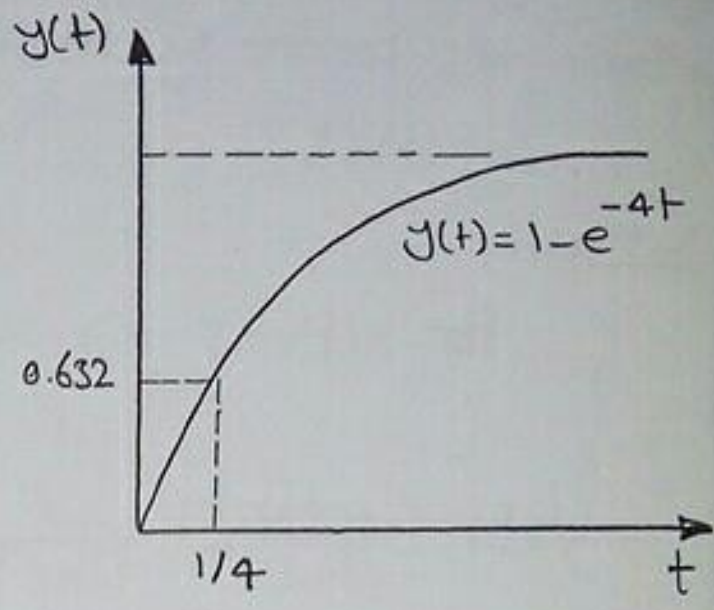
$$K_1 = \lim_{S \to 0} . S . \frac{4}{S(S+4)} = 1$$

$$K_2 = \lim_{S \to -4} (S+4) . \frac{4}{S(S+4)} = -1$$

$$Y(S) = \frac{1}{S} - \frac{1}{S+4}$$

$$y(t) = L^{-1}\left(\frac{1}{S} - \frac{1}{S+4}\right) = 1 - e^{-4t}$$

at $t = \frac{1}{4} \Rightarrow y(t) = 0.632$.



Example: In first-order control system, when input is a step function of 5, the experimentally response is described by:

$$y(t) = 4.5(1 - e^{-12t}).$$

Determine the closed loop transfer function, relating output y to input r.

Solution:

$r(t) = 5$; step function

$\therefore \; r(S) = \frac{5}{S}$

$\therefore Y(S) = r(S) . G(S)$ (because $G(S) = \frac{Y(S)}{r(S)}$

$\Rightarrow y(t) = L^{-1}\left(\frac{5\,G(S)}{S}\right)$

$\because y(t) = 4.5(1 - e^{-12t}) \Rightarrow 4.5(1 - e^{-12t}) = L^{-1}\left(\frac{5\,G(S)}{S}\right)$

$\therefore$ Transformation $\Rightarrow 4.5\left(\frac{1}{S} - \frac{1}{S+12}\right) = \frac{5\,G(S)}{S}$

$$\therefore G(S) = \frac{4.5\,S}{5}\left(\frac{1}{S} - \frac{1}{S+12}\right) = \frac{4.5\,S}{5}\left(\frac{(S+12) - S}{S(S+12)}\right)$$

$$= 0.9\left(\frac{12}{S+12}\right) = \frac{10.8}{S+12}$$

- Response of Second-Order Systems.

Example: For the system with $G(S) = \dfrac{15S}{S^2+8S+15}$, find it's response when $x(t)=3$.

Solution:

For $x(t)=3 \Rightarrow X(S) = \dfrac{3}{S}$

$$\therefore Y(S) = G(S).X(S) = \frac{15\cancel{S}}{S^2+8S+15}\cdot\frac{3}{\cancel{S}} = \frac{45}{(S+5)(S+3)}$$

$$\therefore Y(S) = \frac{K_1}{S+5} + \frac{K_2}{S+3}$$

$$K_1 = \lim_{S \to -5} (S+5)\cdot\frac{45}{(S+5)(S+3)} = \frac{45}{-2} = -22.5$$

$$K_2 = \lim_{S \to -3} (S+3)\cdot\frac{45}{(S+5)(S+3)} = \frac{45}{2} = 22.5$$

$$\therefore Y(S) = \frac{-22.5}{(S+5)} + \frac{22.5}{(S+3)}$$

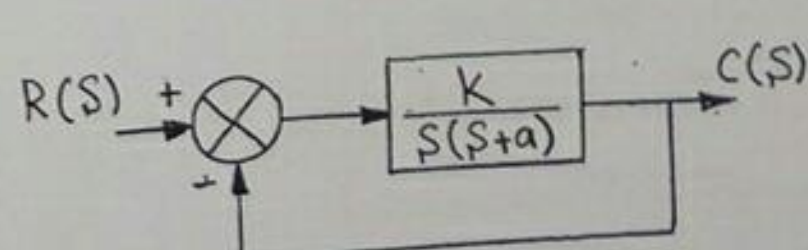
$$\therefore y(t) = L^{-1}\left(\frac{-22.5}{(S+5)} + \frac{22.5}{(S+3)}\right) = -22.5e^{-5t} + 22.5e^{-3t}$$

Example: For a unity feedback system whose open-loop transfer function $G(S) = \dfrac{K}{S(S+a)}$. Determine the values of K and a so that the response to a unit-impulse has the form:

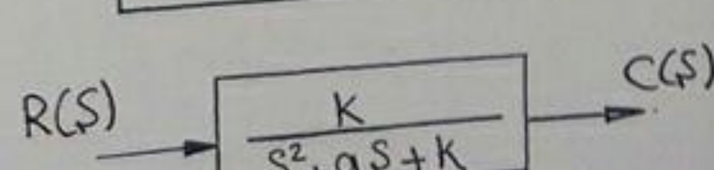
$$c(t) = C_1 e^{-t} + C_2 e^{-4t}$$

when all initial conditions are zero. evaluate C_1 and C_2

Solution:



$R(S)=1$; impulse input



$$\frac{C(S)}{R(S)} = \frac{K}{S^2+aS+K}$$

$$\therefore c(t) = C_1.e^{-t} + C_2.e^{-4t}$$

$$\Rightarrow C(S) = \frac{c_1}{(S+1)} + \frac{c_2}{(S+4)}$$

thus, we can compare ;

$$S^2 + aS + K = (S+1)(S+4)$$

$$\Rightarrow K = 4 \quad \text{and} \quad a = 5$$

$$\therefore C(S) = \frac{4}{S^2+5S+4} = \frac{4}{(S+1)(S+4)} = \frac{c_1}{(S+1)} + \frac{c_2}{(S+4)}$$

$$\Rightarrow c_1 = \lim_{S \to -1} \cancel{(S+1)} \cdot \frac{4}{\cancel{(S+1)}(S+4)} = \frac{4}{3}$$

$$c_2 = \lim_{S \to -4} \cancel{(S+4)} \cdot \frac{4}{(S+1)\cancel{(S+4)}} = -\frac{4}{3}$$

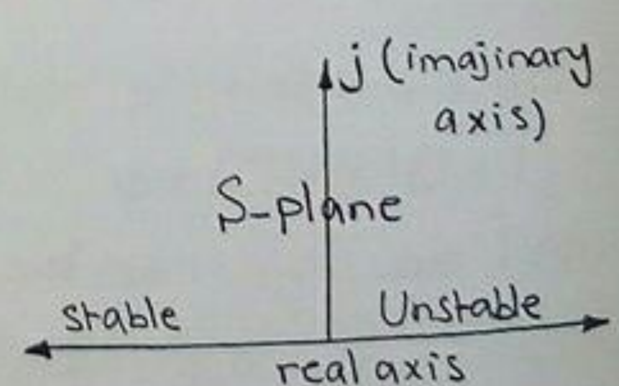
$$\therefore C(S) = \frac{4/3}{S+1} - \frac{4/3}{S+4} \Rightarrow c(t) = \frac{4}{3} e^{-t} - \frac{4}{3} e^{-4t}$$

3.3.5 S-Plane Representation.

It is a method for representing control system transfer functions in the complex plane. This representation is useful in understanding systems performance and stability.

Example: For the system have $G(S) = \dfrac{15}{S^2+8S+15}$ use S-plane to show its stability.

Solution:

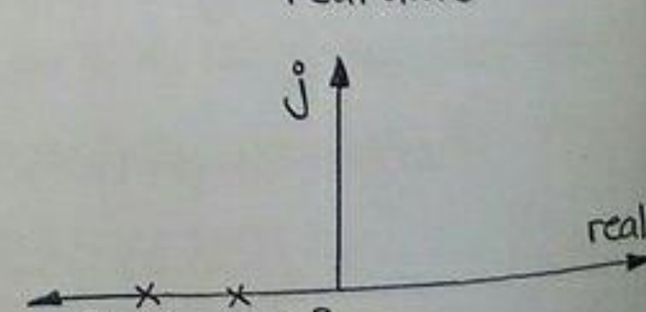


The characteristic equation of the system is :

$$S^2 + 8S + 15 = 0$$

$$\Rightarrow (S+5)(S+3) = 0$$

$$\therefore S_1 = -5 \quad S_2 = -3$$



$\therefore$ the roots are in the left, so the system is stable. (Key point)

Example: For the system $G(S) = \dfrac{10}{S^2+2S-15}$, use S-plane to show If it is stable or no.

Solution:

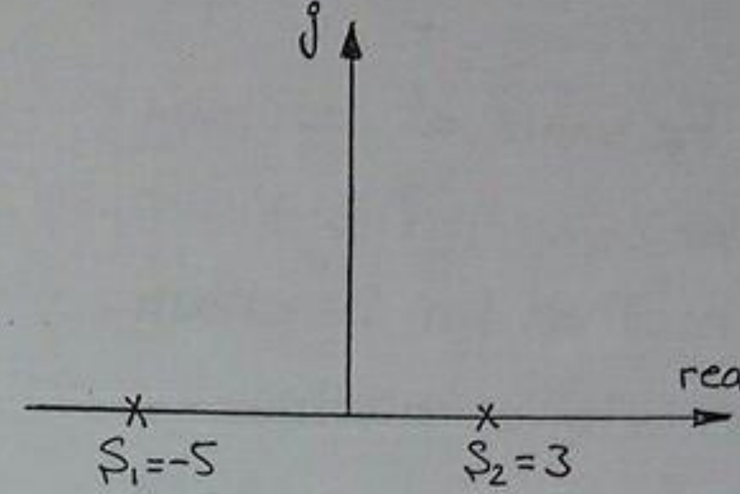
The characteristic equation is;

$$S^2+2S+5=0$$

$$(S+5)(S+3)=0$$

$$\Rightarrow \quad S_1=-5 \quad S_2=3$$

$\therefore$ there is one root in the right half, so the system is unstable.



3.3.6 The General Second Order Control Systems.

Consider a second-order differential equation;

$$a.\frac{d^2y}{dt^2} + b.\frac{dy}{dt} + c.y = e.x(t)$$

take Laplace transforms, with zero initial conditions:

$$a.S^2.Y(S) + b.S.Y(S) + c.Y(S) = e\,X(S)$$

the transfer function is;

$$G(S) = \frac{Y(S)}{X(S)} = \frac{e}{aS^2+bS+c}$$

by $\div c$

$$\Rightarrow G(S) = \frac{e/c}{\frac{a}{c}S^2+\frac{b}{c}S+1}$$

which is Writen as;

$$G(S) = \frac{K}{\frac{1}{\omega_n^2}.S^2+\frac{2\xi}{\omega_n}S+1}$$

which can be normalized to give ;

$$\boxed{G(S) = \frac{K.\omega_n^2}{S^2 + 2\zeta\omega_n.S + \omega_n^2}}$$

which is a standard form of transfer function for a second order system;
where ;

$K \equiv$ steady state gain constant
$\omega_n \equiv$ undamped natural frequency (rad/sec)
$\zeta \equiv$ damping ratio
$\omega_d = \omega_n.\sqrt{1-\zeta^2}$ damped natural frequency when $0<\zeta<1$.

now, it is clear that the characteristic equation of the system is;

$$S^2 + 2\zeta\omega_n.S + \omega_n^2 = 0$$

the roots of this second order equation are ;

$$S_{1,2} = \frac{-2\zeta\omega_n \mp \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2}$$

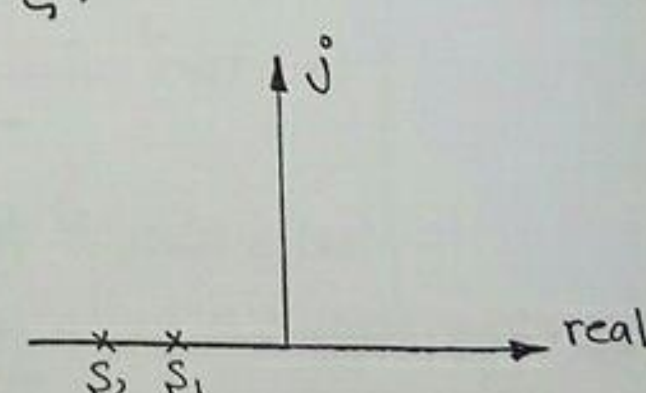
or

$$S_1, S_2 = -\zeta.\omega_n \mp \omega_n\sqrt{\zeta^2-1}$$

i) Over-damped transient response ($\zeta > 1$)

∴ the roots are ;

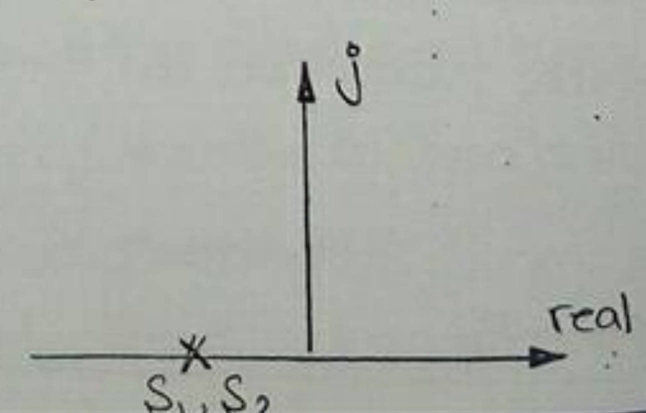
$$S_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$$
$$S_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$$



ii) Critically damped transient response ($\zeta = 1$)

∴ the roots are ;

$$S_1 = S_2 = -\omega_n$$

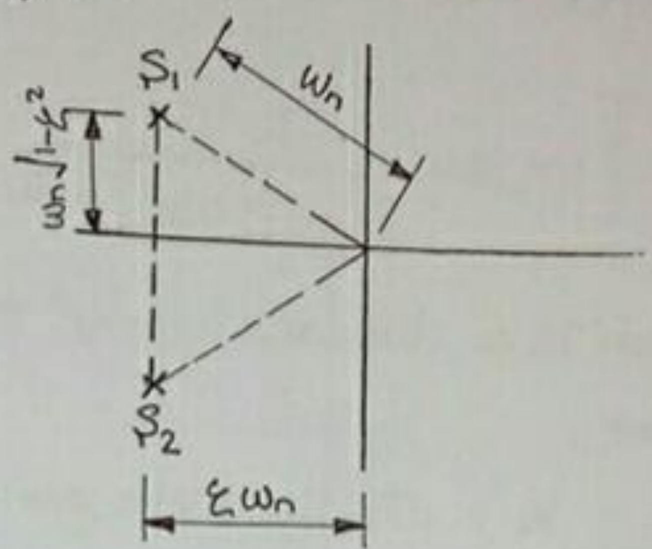


iii) Un-damped transient response ($\zeta < 1$)

the roots are :

$$S_1 = -\zeta \omega_n + j\omega_n\sqrt{1-\zeta^2}$$

$$S_2 = -\zeta \omega_n - j\omega_n\sqrt{1-\zeta^2}$$



For a unit step function input of $X(S) = \frac{1}{S}$, the output response is;

$$Y(S) = G(S) \cdot X(S)$$

$$\Rightarrow \quad Y(S) = \frac{K \cdot \omega_n^2}{S(S^2 + 2\zeta\omega_n S + \omega_n^2)}$$

and for constant gain K=1 and undamped response ($\zeta<1$), the output is;

$$Y(S) = \frac{\omega_n^2}{S(S^2 + 2\zeta\omega_n S + \omega_n^2)}$$

$$\therefore \boxed{y(t) = 1 \mp \frac{1}{\sqrt{1-\zeta^2}} \cdot e^{-\zeta\omega_n t} \cdot \sin(\omega_d \cdot t + \theta)} \qquad (a)$$

where :

$$\boxed{\begin{aligned} \theta &= \tan^{-1}\frac{-\sqrt{1-\zeta^2}}{-\zeta} \\ \omega_d &= \omega_n\sqrt{1-\zeta^2} \end{aligned}}$$

when $\zeta = 1 \Rightarrow$

$$y(t) = 1 - e^{-\omega_n \cdot t} \cdot (1 + \omega_n t)$$

*Note : Equation (a) can be writen in the form :

$$y(t) = 1 - e^{-\zeta\omega_n \cdot t} \cdot \left(\cos\omega_d \cdot t + \frac{\zeta}{\sqrt{1-\zeta^2}} \cdot \sin\omega_d \cdot t\right)$$

Example: A system with transfer function $G(S) = \frac{100}{S^2+10S+100}$, find :-

1. ω_n and ζ for the system
2. Unit step response

Solution:-

System characterestic equation is:

$$S^2+10S+100=0$$

which can be compared with;

$$S^2+2\zeta\omega_n S+\omega_n^2=0$$

$\therefore \omega_n^2=100 \quad \Rightarrow \omega_n=10$

$2\zeta\omega_n=10 \quad \Rightarrow \because \omega_n=10 \quad \therefore \zeta=0.5$

for unit step response where $\zeta=0.5<1$, therefore;

$$y(t)=1\mp\frac{1}{\sqrt{1-\zeta^2}}.e^{-\zeta\omega_n.t}.\sin(\omega_d.t+\theta)$$

$$\omega_d=\omega_n.\sqrt{1-\zeta^2}=8.66$$

$$\theta=\tan^{-1}\frac{-\sqrt{1-\zeta^2}}{-\zeta}=60°$$

$\therefore y(t)=1\mp1.154\,e^{-5t}.\sin(8.66t+60°)$

3.4 Definitions of Transient Response Specification.

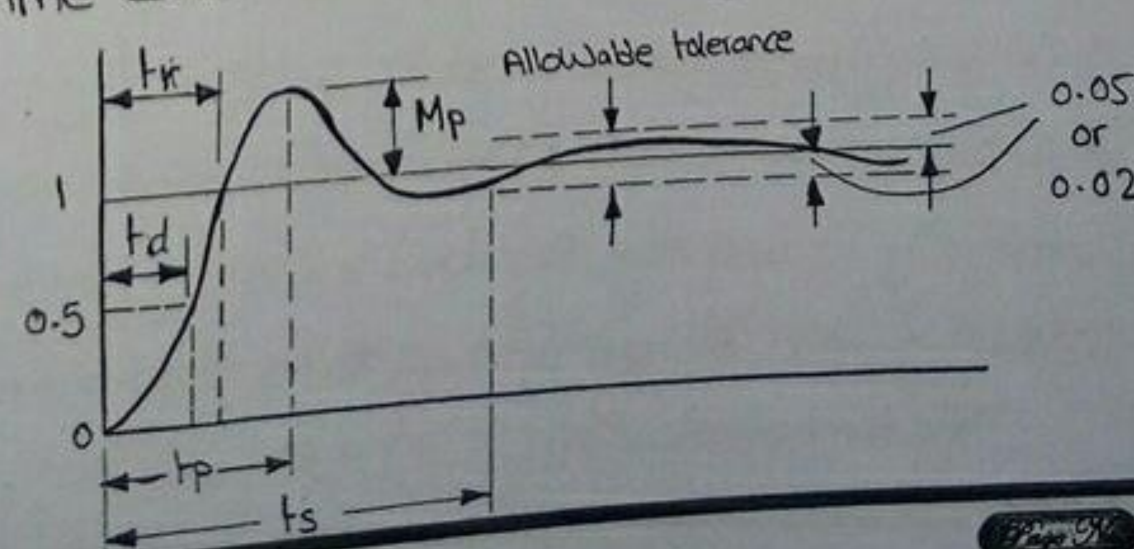
Frequently, the performance characteristics of a control system are specified in terms of the transient response to a unit-step input, since it is easy to generate and is sufficiently drastic. The transient response of a system to a unit-step input depends on the initial conditions. It is commolly to use the standard initial conditions that the system is at rest initially with output and all time derivatives thereof zero.

The transient response of a ~~par~~ practical control system often exhibits damped oscillations before reaching steady state. In specifying the transient response characteristics of a control system to a unit-step input, it is common to specify the following:

1. Delay time (t_d): The delay time is the time required for the response to reach half the final value the very first time.
2. Rise time (t_r): The rise time is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value. For underdamped second order systems, the 0% to 100% rise time is normally used. For overdamped systems, the 10% to 90% rise time is commonly used.
3. Peak time (t_p): The peak time is the time required for the response to reach the first peak of the overshoot.
4. Maximum (percent) overshoot (M_p): The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by:

$$\text{Maximum percent overshoot} = \frac{y(t_p) - y(\infty)}{y(\infty)} \times 100\%$$

5. Settling time (t_s): The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%). The settling time is related to the largest time constant of the control system.



3.5 Second-Order Systems and Transient Response Specifications.

In the following, we shall obtain the rise time, peak time, maximum overshoot, and settling time of the second-order system given by the equation;

$$\boxed{\frac{Y(S)}{X(S)} = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n . S + \omega_n^2}}$$

These values will be obtained in terms of ζ and ω_n. The system is assumed to be underdamped.

$$\therefore \boxed{y(t) = 1 + \frac{e^{-\zeta\omega_n . t}}{\sqrt{1-\zeta^2}} \sin(\omega_d . t + \theta)}$$

- For "Rise Time" (t_r):

We obtain the rise time (t_r) by letting $y(t_r) = 1$

$$\therefore y(t_r) = 1 = 1 + \frac{e^{-\zeta\omega_n . t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d . t_r + \theta).$$

$$\Rightarrow e^{-\zeta\omega_n . t_r} . \sin(\omega_d . t_r + \theta) = 0$$

since $e^{-\zeta\omega_n . t_r} \neq 0$ $\quad \therefore \sin(\omega_d . t_r + \theta) = 0$

$\Rightarrow \omega_d . t_r + \theta = n\pi \quad$ where $n = 1, 2, 3, \dots$

$$\therefore \boxed{t_r = \frac{\pi - \theta}{\omega_d}}$$

- For "Peak Time" (t_p):

Peak time (t_p) may be obtained by differentiating $y(t)$ with respect to time and equal zero.

$$\frac{dy(t)}{dt} = \frac{\omega_n . e^{-\zeta\omega_n . t}}{\sqrt{1-\zeta^2}} \sin \omega_d . t_p$$

now, let $\frac{dy(t)}{dt} = 0$

$\therefore$ $\sin \omega_d . t_p = 0 \Rightarrow \omega_d . t_p = n\pi$ where $n = 1, 2, 3, \ldots$

$$\therefore \quad \boxed{t_p = \frac{n\pi}{\omega_d}}$$

the maximum value of $y(t)$ occurs when ;

$$\boxed{t_p = \frac{\pi}{\omega_d}}$$

- For "Maximum overshoot" (M_p):

The maximum overshoot occurs at the peak time or at $t = t_p = \frac{\pi}{\omega_d}$. Assuming the final value of the output is unity, then "M_p" can be obtained from ;

$$M_p = y(t_p) - 1$$
$$= 1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n\left(\frac{\pi}{\omega_d}\right)} . \sin\left(\omega_d . \frac{\pi}{\omega_d} + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right) - 1$$

$$\Rightarrow \quad \boxed{M_p = e^{-\pi(\xi/\sqrt{1-\xi^2})}}$$

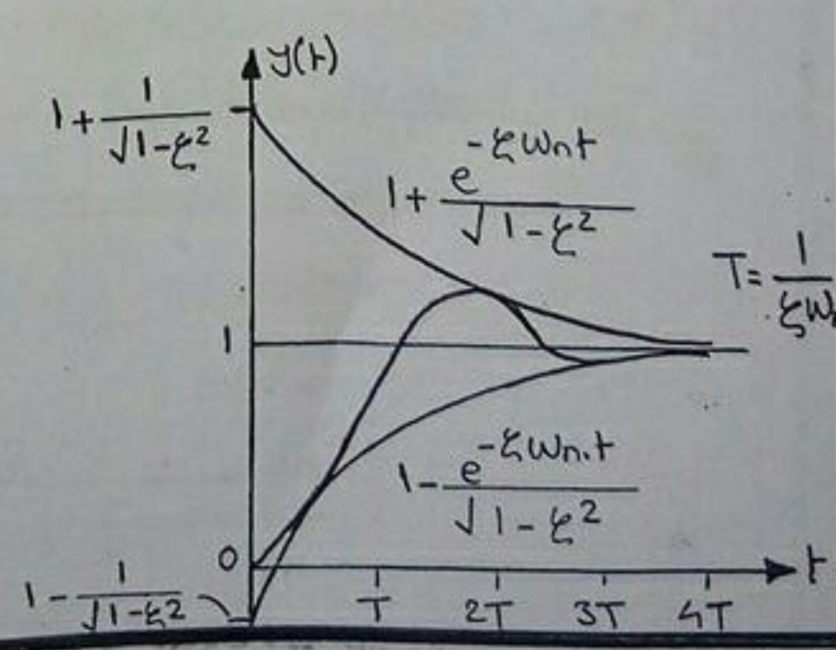
and the maximum percent overshoot is;

$$\boxed{M_p = e^{-\pi(\xi/\sqrt{1-\xi^2})} \times 100\%}$$

- For "Setting Time" (t_s):

The setting time (t_s) occurs when the equations of ~~the~~ envelope are evaluated with certain percentage (5%) and (2%) of its final value.

In the figure given here,



the curves $1 \pm (e^{-\xi\omega_n.t} / \sqrt{1-\xi^2})$ are the envelope curves of the translent response to a unit-step input. The response curve y(t) always remains within a pair of the envelope curves. As it can be seen, the time constant of these envelope curves is $1/\xi\omega_n$.

It have been found that the settling time reaches a minimum value around $\xi = 0.76$ for the 2% eriterion or $\xi = 0.68$ for the 5% criterion. As well as, if the 2% criterion is used, (t_s) is approximatelly four times the time constant of the system. If the 5% criterion is used, (t_s) is approximatelly three times the time constant. See the following:

for 5% percentage of the final value of the upper envelope:

$$y(t) = 1 + \frac{e^{-\xi\omega_n.t_s}}{\sqrt{1-\xi^2}} = 1.05$$

$$\Rightarrow \therefore \omega_n.t_s = -\frac{1}{\xi}.\ln(0.05\sqrt{1-\xi^2})$$

$\therefore$ for a very small value of ξ, we get:

$$\omega_n.t_s \simeq \frac{3}{\xi} \quad \Rightarrow t_s = \frac{3}{\xi\omega_n} \quad \text{for 5\% criterion}$$

or

$$\omega_n.t_s = \frac{4}{\xi} \quad \Rightarrow t_s = \frac{4}{\xi\omega_n} \quad \text{for 2\% criterion}$$

For Impulse response of second order systems, the input is unity $X(S)=1$. The unit-impulse response is:

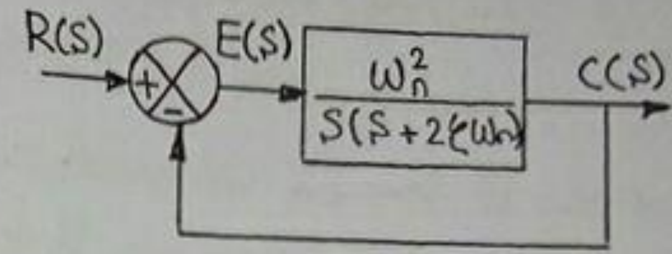
$$Y(S) = \frac{\omega_n^2}{S^2 + 2\xi\omega_n.S + \omega_n^2}$$

$\therefore$ the inverse Laplace transform of this equation when $\xi < 1$.

$$y(t) = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n.t}.\sin(\omega_n\sqrt{1-\xi^2}.t)$$

Example: Consider the system show in the figure, where $\zeta = 0.6$ and $\omega_n = 5$ rad/sec. Let us obtain the rise time, peak time, maximum overshoot and settling time when the system is subjected to a unit-step input.

Solution:



From the given values of ζ and ω_n,

we obtain $\omega_d = \omega_n\sqrt{1-\zeta^2} = 4$

$$\sigma = \zeta \cdot \omega_n = 3$$

Rise time (t_r):

$$t_r = \frac{\pi - \theta}{\omega_d}$$

but here θ is given by $\theta = \tan^{-1}\frac{\omega_d}{\sigma}$

$\therefore \theta = \tan^{-1}\frac{4}{3} = 0.93$ rad.

$$\therefore t_r = \frac{\pi - 0.93}{4} = 0.55 \text{ sec}$$

Peak time (t_p):

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{4} = 0.785 \text{ sec.}$$

Maximum overshoot (M_p):

$$M_p = e^{-(\sigma/\omega_d)\pi} = e^{-(3/4)\pi} = 0.095$$

$\therefore$ the maximum percent overshoot is therefore 9.5 %.

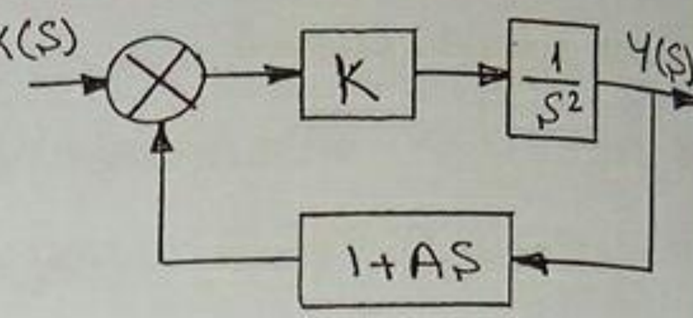
Settling time (t_s):

for the 2% criterion $t_s = \frac{4}{\sigma} = \frac{4}{3} = 1.33$ sec

for the 5% criterion $t_s = \frac{3}{\sigma} = \frac{3}{3} = 1$ sec.

Example : Consider the servomechanism shown in the figure, determine the values of A and K sothat the maximum overshoot in unit-step response is 25% and peak time is 2 sec.

Solution :



$\because M_p = 100\, e^{-\zeta\pi/\sqrt{1-\zeta^2}}$

$25 = 100\, e^{-\zeta\pi/\sqrt{1-\zeta^2}}$

$\Rightarrow \dfrac{\zeta\pi}{\sqrt{1-\zeta^2}} = 1.39 \quad \Rightarrow \zeta = 0.4$

Also, $t_p = \dfrac{\pi}{\omega_d} \Rightarrow 2 = \dfrac{\pi}{\omega_d} \quad \Rightarrow \therefore \omega_d = 1.57$ rad/sec.

$\because \omega_d = \omega_n\sqrt{1-\zeta^2} \quad \Rightarrow \omega_n = 1.713$ rad/sec.

We can get from the block diagram;

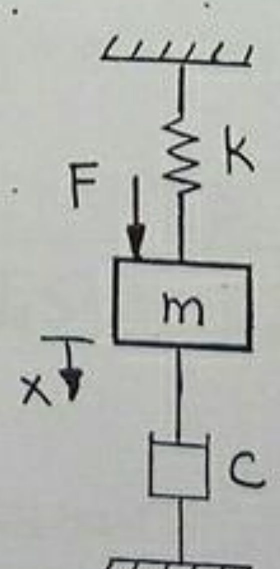
$\dfrac{Y(S)}{X(S)} = \dfrac{K}{S^2 + KAS + K}$ comparing with $\dfrac{Y(S)}{X(S)} = \dfrac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}$

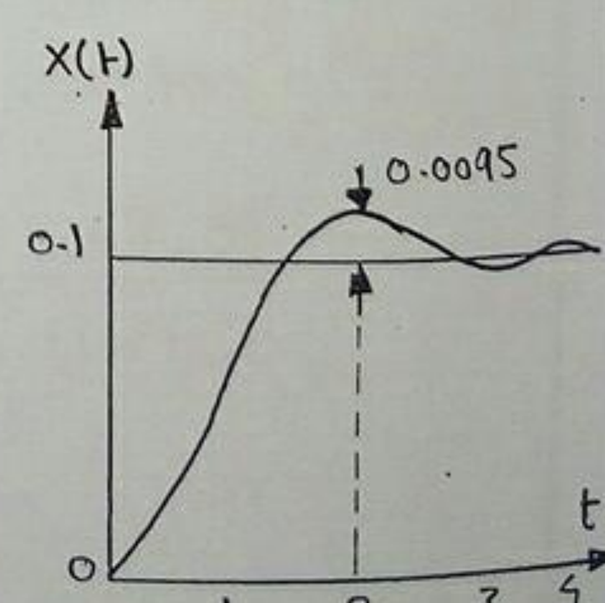
$\Rightarrow \therefore \omega_n^2 = K \quad \Rightarrow K = (1.713)^2 = 2.9$

$2\zeta\omega_n = KA \Rightarrow A = \dfrac{2\zeta\omega_n}{K} = \dfrac{2\times 0.4 \times 1.71}{2.9} = 0.47.$

Example: A mechanical vibratory system with applied force of 2N is used. The mass oscilates as shown. Determine m, C and K of the system from the response curve given.

Solution:





$\Sigma F = m\ddot{x}$

$\Rightarrow F - Kx - C\dot{x} = m\ddot{x}$

$\therefore m\ddot{x} + C\dot{x} + Kx = F$

$(mD^2 + CD + K)x = F$

Laplace transform of this equation;

$(mS^2 + CS + K)\,X(S) = F(S)$

$$\Rightarrow \frac{X(S)}{F(S)} = \frac{1}{mS^2 + CS + K}$$

$\because F = 2\ N \Rightarrow F(S) = \frac{2}{S}$

$$\therefore X(S) = \frac{2}{S(mS^2 + CS + K)}$$

the steady state value of x is ;

$$X(\infty) = \lim_{S \to 0} S.X(S) = \frac{2}{K} = 0.1 \quad \Rightarrow K = 20\ N/m$$

$$M_p = \frac{0.0095}{0.1} = 9.5\%$$

$$\Rightarrow 9.5 = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \quad \Rightarrow \zeta = 0.6$$

$\because t_p = 2\ sec \Rightarrow \omega_n = 1.96\ rad/sec.$

however, $\omega_n^2 = K/m \quad \Rightarrow m = 5.2\ kg.$

$$2\zeta\omega_n = \frac{C}{m} \quad \Rightarrow C = 12.2\ N/m.sec.$$

4.1 Steady-State Errors in Control Systems.

Error in a control system can be attributed to many factors. Changes in the reference input will cause unavoidable errors during transient periods and may also cause steady-state errors. Imperfictions in the system components, such as static friction, backlash, and amplifier drift, as well as aging or deterioration, will cause errors at steady state.

Any physical control system inherently suffers steady-state error in response to certain types of inputs. A system may have no steady-state error to a step input, but the same system may exhibit non-zero steady-state error to a ramp input. The only way we may be able to eliminate this error is to modify the system structure.

4.2 Classification of Control Systems.

Control systems may be classified according to their ability to follow step inputs, ramp inputs, parabolic input, and so on. This is a reasonable classification scheme, because actual inputs may frequently be considered combinations of such inputs. The magnitudes of the steady-state errors due to these individual inputs are indicative of the goodness of the system.

We can establish the type of control system by refering to the form of $G(S).H(S)$. The loop-system may be written as:

$$G(S).H(S) = \frac{K(1+T_1S)(1+T_2S)\cdots(1+T_mS)}{S^1(1+T_aS)(1+T_bS)\cdots(1+T_nS)} \quad (1)$$

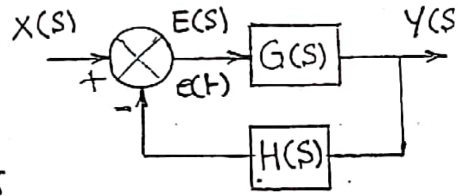
Where: K and T are constant.

It involves the term S^i in the denominator, representing a pole of multiplicity (i) at the origin. The present classification scheme is based on the number of integrations indicated by the open-loop transfer function. A system is called type 0, type 1, type 2,.. if $N=0$, $N=1$, $N=2$, ..., respectively. As the type number is increased, accuracy is improved; however, increasing the type number aggravates the stability problem. A compromise between steady-state accuracy and relative stability is always necessary.

4.3 Steady-State Errors.

It is a measure of the control system accuracy in tracking a command input or in rejecting a disturbance in the form of a load change. The steady-state errors of control systems depend on the input and the type of the system. The steady-state errors is defined from closed loop system as:

$$\frac{Y(S)}{X(S)} = \frac{G(S)}{1+G(S).H(S)}$$



the transfer function between the error signal $e(t)$ and the input signal $x(t)$ is:

$$\frac{E(S)}{X(S)} = 1 - \frac{Y(S).H(S)}{X(S)} = \frac{1}{1+G(S).H(S)}$$

where the error $e(t)$ is the difference between the input signals.

The final value theorm provides a convenient way to find the steady-state performance of a stable system. Since $E(S)$ is:

$$E(S) = \frac{1}{1+G(S).H(S)} . X(S)$$

the steady-state error is :

$$\boxed{e_{ss} = \lim_{t \to \infty} e(t) = \lim_{S \to 0} S.E(S) = \lim_{S \to 0} . \frac{S\, X(S)}{1+G(S)H(S)}}$$

Where :

$e_{ss} \equiv$ Steady-State errors

$e(t) \equiv$ error signal response

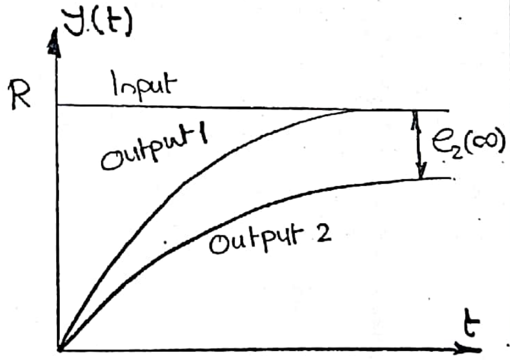
4.4 Classification of Steady-State Errors Depending on Input Signals.

4.4.1 Steady-State Error Due to a Step Input.

If the reference input to the control system is a step input of magnitude (R), the laplace transform $X(S) = \frac{R}{S}$

$$e_{ss} = e(\infty) = \lim_{t \to \infty} e(t) = \lim_{S \to 0} S.E(S)$$

$$E(S) = \frac{X(S)}{1+G(S)H(S)}$$



$$X(S) = \frac{R}{S} \quad \text{for step input}$$

$$e_{ss} = \lim_{S \to 0} . S . \frac{R/S}{1+G(S)H(S)}$$

$$= \lim_{S \to 0} \frac{R}{1+G(S)H(S)}$$

∴ for unit step input :

$$e_{ss} = \lim_{S \to 0} \frac{1}{1+G(S)H(S)} \quad \Rightarrow e_{ss} = \frac{1}{1+K_p}$$

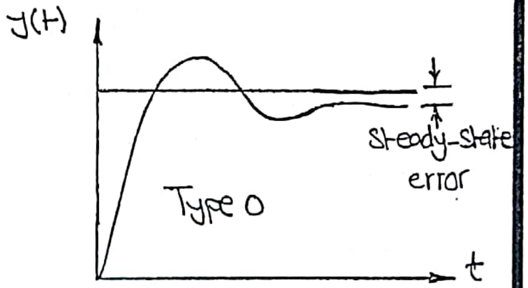
where $K_p = \lim_{S \to 0} G(S)H(S)$

K_p is called the positional error constant. From the figure, the output response of two types the first has a zero steady state error, and the second has a finite steady state error $e_2(\infty)$. For control systems, we have:

a) Type 0 System:

For a type 0 we have $i=0$, substituting in equation (1) and $\lim_{S\to 0} G(S)H(S)=K$, then;

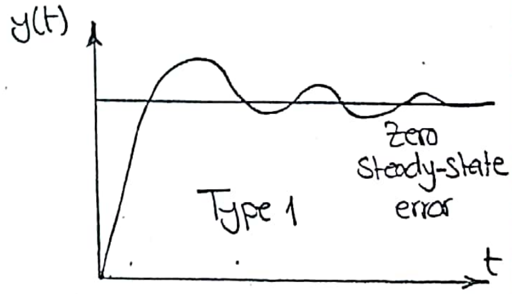
$$e_{ss} = \frac{R}{1+K_p} = \text{constant}$$



b) Type 1 System:

For a type 1 we have $i=1$, Substituting in equation (1) and $\lim_{S\to 0} G(S).H(S) = \infty$;

$$e_{ss} = \frac{R}{1+\infty} = 0$$



C) Type 2 System:

For a type 2 we have $i=2$, substituting in equation (1) and $\lim_{S\to 0} G(S)H(S)=\infty$;

$$e_{ss}=0$$

4.4.2 Steady-State Errors Due to a Ramp Input.

If the input signal to a control system is $x(t)=R.t$, the laplace transform is,

$$X(S) = \frac{R}{S^2}$$

c) Type 2 system :

For a type 2 we have $i=2$, substituting in equation (1) we get $G(S)H(S) = \dfrac{R}{S^2(1+\frac{K}{S^2})}$

$$\Rightarrow \therefore \overset{K_v}{\cancel{e_{ss}}} = \lim_{S\to 0} S.G(S)H(S) \Rightarrow \lim_{S\to 0} \frac{R}{S(1+\frac{K}{S^2})} = e_{ss}$$

$$\Rightarrow \therefore e_{ss} = 0$$

4.4.3. Steady-State Error Due to a Parabolic Input.

If the input $x(t) = \frac{1}{2}R.t^2$, the laplace transform :

$$X(S) = \frac{R}{S^3}$$

$$e_{ss} = \lim_{S\to 0} S.\frac{R/S^3}{1+G(S)H(S)}$$

$$= \lim_{S\to 0} \frac{R}{S^2 G(S)H(S)}$$

$$\Rightarrow e_{ss} = \frac{R}{K_a}$$

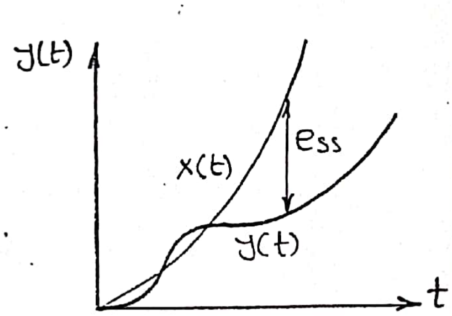
where $K_a = \lim_{S\to 0} S^2.G(S)H(S)$ ≡ Parabolic error constant.

a) Type 0 System :

For a type 0 we have $i=1$, sub. in equation (1) we get ;

$G(S)H(S) = K$

$$\therefore e_{ss} = \frac{R}{K_a}$$



Type 0 and 1

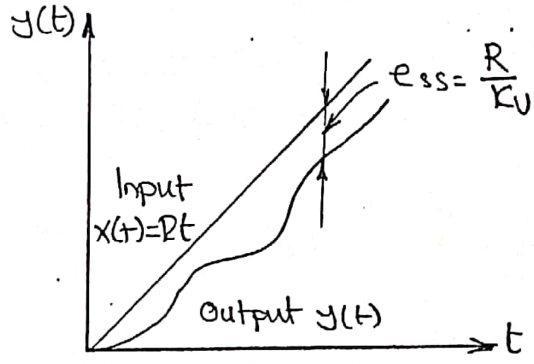
and however, $K_a = \lim_{S\to 0} S^2.G(S)H(S)$

$$\Rightarrow e_{ss} = \infty$$

$$e_{ss} = \lim_{S \to 0} . S . \frac{R/S^2}{1+G(S)H(S)}$$

$$= \lim_{S \to 0} . \frac{R}{S + S\,G(S)H(S)}$$

$$= \frac{R}{\lim_{S \to 0} . S\,G(S)H(S)}$$



$$\therefore e_{ss} = \frac{R}{K_v}$$

where $K_v = \lim_{S \to 0} S.G(S).H(S) \equiv$ ramp error constant.

a) Type 0 System :

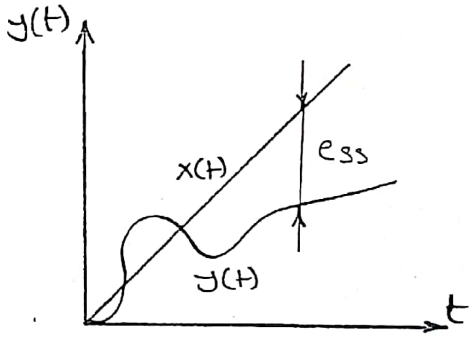
For a type 0 we have $i=0$, and substituting in equation (1) we obtain ;

$$G(S).H(S) = K$$

$$\therefore e_{ss} = \frac{R}{K_v}$$

and however, $K_v = \lim_{S \to 0} . S.K = 0$

$$\therefore e_{ss} = \frac{R}{0} = \infty$$



Type 0

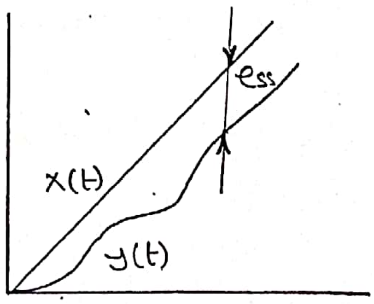
b) Type 1 system :

For a type 1 we have $i=1$, and substituting in equation (1) we get ;

$$G(S).H(S) = 1 + \frac{K}{S}$$

$$\therefore K_v = \lim_{S \to 0} . S.(1 + \frac{K}{S})$$

$$\therefore e_{ss} = \frac{R}{K_v} = \text{constant}$$



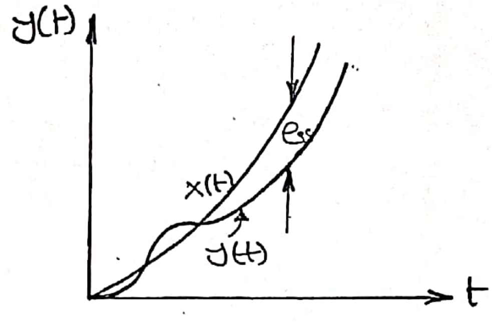
Type 1

b) Type 1 system :

For a type 1 we have $i=1$, and sub. in equation (1) we get;

$$G(S)H(S) = 1 + \frac{K}{S}$$

$\therefore e_{ss} = \infty$



Type 2

c) Type 2 system :

For a type 2 we have $i=2$, and sub. in equation (1) we get;

$$e_{ss} = \frac{R}{K_a} = \text{constant.}$$

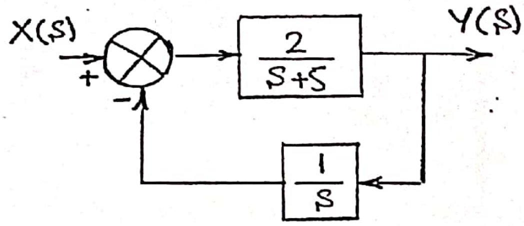
Summary of the Steady-State Error due to step, ramp, and parabolic Inputs.

Type of System (i)	K_p	K_v	K_a	Step Input $e_{ss} = \frac{R}{1+K_p}$	Ramp Input $e_{ss} = \frac{R}{K_v}$	Parabolic Input $e_{ss} = \frac{R}{K_a}$
0	K	0	0	$e_{ss} = \frac{R}{1+K}$	$e_{ss} = \infty$	$e_{ss} = \infty$
1	∞	K	0	$e_{ss} = 0$	$e_{ss} = \frac{R}{K}$	$e_{ss} = \infty$
2	∞	∞	K	$e_{ss} = 0$	$e_{ss} = 0$	$e_{ss} = \frac{R}{K}$

Example: For the system shown, find the steady-state error for unit step, unit ramp and unit parabolic inputs.

Solution:

$$G(S).H(S) = \frac{2}{S(S+5)}$$



So the system is from type 1

$\Rightarrow \therefore K_p = \infty \Rightarrow e_{ss} = 0$

$$K_v = K = \lim_{S \to 0} S \cdot \frac{2}{S(S+5)} = \frac{2}{5}$$

$$\therefore e_{ss} = \frac{1}{K_v} = \frac{5}{2} \quad (\text{constant})$$

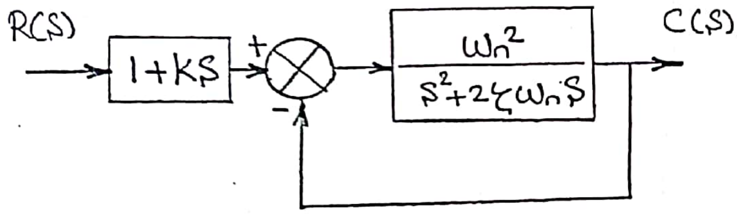
$K_a = 0 \Rightarrow e_{ss} = \infty$

Example: The control system represented by the shown figure, has a unit-ramp input. Obtain the value of (K) so that the steady-state error can be eliminated.

Solution:

For unit ramp input;

$$R(S) = \frac{1}{S^2}$$



$$\frac{C(S)}{R(S)} = \frac{\omega_n^2(1+KS)}{S^2+2\zeta\omega_n S+\omega_n^2}$$

$$C(S) = \frac{\omega_n^2(1+KS)}{S^2(S^2+2\zeta\omega_n S+\omega_n^2)}$$

$$E(S) = R(S) - C(S)$$

$$\therefore E(S) = \frac{1}{S^2} - \frac{\omega_n^2(1+KS)}{S^2(S^2+2\zeta\omega_n S+\omega_n^2)} = \frac{S^2+2\zeta\omega_n S - K\omega_n^2 S}{S^2(S^2+2\zeta\omega_n+\omega_n^2)}$$

$$\therefore e_{ss} = \lim_{S \to 0} . S . E(S)$$

$$\therefore e_{ss} = \lim_{S \to 0} . S . \left[\frac{S(S + 2\xi\omega_n - K\omega_n^2)}{S^2(S^2 + 2\xi\omega_n S + \omega_n^2)} \right]$$

$$= \lim_{S \to 0} \left[\frac{S + 2\xi\omega_n - K\omega_n^2}{S^2 + 2\xi\omega_n S + \omega_n^2} \right]$$

$$\therefore e_{ss} = \frac{2\xi\omega_n - K\omega_n^2}{\omega_n^2}$$

In order to eliminate the steady-state error, let $e_{ss} = 0$

$$\therefore \frac{2\xi\omega_n - K\omega_n^2}{\omega_n^2} = 0 \qquad \therefore K = \frac{2\xi}{\omega_n}$$

5.1 Routh- Hurwitz Criterion.

The Routh-Hurwitz criterion is an algebraic method that Indicates whether or not all roots of the characteristic equation have negative real parts without actually finding the roots. In this case the stability condition is not satisfied, the method also indicate the number of roots that lie in the right half of the S-plane and on the imaginary axis, that is the number of roots that have positive and zero real parts.

The technique was developed Independently Routh in the 1890 and for this reason, it's also called the Routh-criterion. We have seen earlier that the characteristic equation of a closed-loop control system is given by:

$$a_n S^n + a_{n-1} S^{n-1} + \dots + a_1 S + a_0 = 0 \qquad (1)$$

where, n is the order of the system, and a_i are constant coefficients. The necessary and sufficient condition that all roots of equation (1) lie in the left half of S-plane, so that all the coefficients of the equation be positive and all terms in the first column of the array have positive signs.

The first step in the simplification of Routh-criterion is to arrange the polynomial coefficients into two rows. The first row consist of the first, third, fifth, ... coefficients, and the second row consists of the second, fourth, sixth, ... coefficients as shown in the following tabulation;

$$\left.\begin{matrix} a_0 & a_2 & a_4 & a_6 & a_8 & \dots \\ a_1 & a_3 & a_5 & a_7 & a_9 & \dots \end{matrix}\right\} \qquad (2)$$

The next step is to form the following array of numbers by the indicated operations (the example shown is for a sixth-order system) :-

$$a_0 S^6 + a_1 S^5 + a_2 S^4 + a_3 S^3 + a_4 S^2 + a_5 S + a_6 = 0$$

S^6	a_0	a_2	a_4	a_6
S^5	a_1	a_3	a_5	0
S^4	$\frac{a_1 a_2 - a_0 a_3}{a_1} = A$	$\frac{a_1 a_4 - a_0 a_5}{a_1} = B$	$\frac{a_1 a_6 - a_0 X_0}{a_1} = a_6$	0
S^3	$\frac{A a_3 - a_1 B}{A} = C$	$\frac{A a_5 - a_1 a_6}{A} = D$	$\frac{A X_0 - a_1 X_0}{A} = 0$	0
S^2	$\frac{BC - AD}{C} = E$	$\frac{C a_6 - A X_0}{C} = a_6$	$\frac{C X_0 - A X_0}{C} = 0$	0
S^1	$\frac{ED - C a_6}{E} = F$	0	0	0
S^0	$\frac{F a_6 - E X_0}{F} = a_6$	0	0	0

The last step is to investigate the signs of the numbers in the first column of the tabulation. The roots of the polynomials are all in the left half of the S-plane if all the elements of the first column of the Routh tabulation are of the same sign. If there are changes of signs in the elements of the first column, the number of sign changes indicates the number of roots with positive real parts.

Example: Consider the equation $2S^4 + S^3 + 3S^2 + 5S + 10 = 0$. Investigate the stability of the control system?

Solution:

S^4	2	3	10
S^3	1	5	0
S^2	$\frac{(1\times3)-(2\times5)}{1}=-7$	10	0
S^1	$\frac{(-7\times5)-(1\times10)}{-7}=\frac{45}{7}$	0	0
S^0	10		

Since there are two changes in sign in the first column, the equation has two roots in right hand half S-plane, such that system is unstable.

Example: Consider the equation $(S-2)(S+1)(S-3)=0$, which has two positive real parts, can be illustrated in Routh-criterion as:

Solution:

$$(S-2)(S+1)(S-3)=S^3-4S^2+S+6=0$$

S^3	1	1
S^2	-4	6
S^1	$\frac{(-4\times1)-(6\times1)}{-4}=2.5$	0
	$\frac{(2.5\times6)-(4\times0)}{2.5}=6$	0

Since there are two sign changes in the first column, means there are two roots in right half of S-plane which agree with the first equation.

5.2 Special Cases

The two Illustrative example given above are designed so that the Routh-criterion can be carried out without any complication. However, depending upon the equation to be tested, the following dif-ficalties may occur occasionally when carring out the Routh test:

1) The first element in any one row of the Routh tabulation is zero, but other elements are not.

2) The elements in one row of the Routh tabulation are all zero.

5.2.1 Case 1:

If a zero appears in the first position of a row, the element in the next row will all become infinite. We replace the zero e-lement in the Routh-tabulation by an arbitrary small positive number (ϵ) and complete the array.

Example: Consider the equation $S^3 - 3S + 2 = 0$

Solution:

$$\begin{array}{c|cc} S^3 & 1 & -3 \\ S^2 & 0 & 2 \\ S^1 & \infty & \end{array}$$

Because of zero in the first element of the second row, the first element of the third row is infinite. To correct this ~~stat~~ situation, we may replace the zero element in the second row of the Routh tabulation by (ϵ).

then We have :

$$
\begin{array}{c|cc}
s^3 & 1 & -3 \\
s^2 & \epsilon & 2 \\
s^1 & \frac{-3\epsilon-2}{\epsilon} & 0 \\
s^0 & 2 &
\end{array}
$$

Since (ϵ) is a small positive number, so there are two changes in sign of the elements of the first column. Thus, the system is unstabile.

Example: Consider the equation $S^5+2S^4+2S^3+4S^2+6S+8=0$

Solution:

$$
\begin{array}{c|ccc}
s^5 & 1 & 2 & 6 \\
s^4 & 2 & 4 & 8 \\
s^3 & \cancel{0}\,\epsilon & 2 & 0 \\
s^2 & c_1 & 8 & 0 \\
s^1 & d_1 & 0 & 0 \\
s^0 & 8 & &
\end{array}
$$

The zero number in the first column is replaced by ϵ and we obtain:

$$c_1 = \frac{4\epsilon-4}{\epsilon}$$

$$d_1 = \frac{2c_1-8\epsilon}{c_1} = \frac{8\epsilon-8-8\epsilon^2}{4\epsilon-4}$$

Since ϵ is a small positive number, it follows that; $c_1<0$ and $d_1 \rightarrow 2$ as $\epsilon \rightarrow 0$. As a result, there are two changes in sign indicating unstable system.

5.2.2 Case 2 :

There is zero in the first column and all other elements of that row are zero. It indicates that one or more of the following conditions may exist :

1) Pairs of real roots with opposite signs.
2) Pairs of imaginary roots.
3) Pairs of complex-conjugate roots forming symmetry about the origin of the S-plane.

The equation that is formed by using the coefficients of the row just above the row of zeros is called the auxilliary equation. The order of the auxilliary equation is always even, indicating the number of the root pairs that are equal in magnitude but opposite in sign. The auxilliary equation with second order refers to two equal and opposite roots. All these roots of equal magnitude can be obtained by solving the auxilary equation. When a row of zeros appears in the Routh tabulation again the test breaks down.

** Method (1) : The test may be carried on by performing the following remedies :

1. Take the derivative of the auxilliary equation with respect to S.
2. Replace the row of zeros with the coefficients of the resultant equation obtained by taking the derivative of the auxilliary equation.
3. Carry on the Routh test in the usual manner with the newly formed tabulation.

Example: Consider the equation $S^4+S^3-3S^2-S+2=0$.

Solution:

S^4	1	-3	2
S^3	1	-1	0
S^2	-2	2	0
S^1	0	0	0
S^0			

since the S^1 row contains all zeros, we form the auxillbary equation using the coefficients contained in the S^2 row. Thus the auxilliary equation is written as:

$$A(S) = -2S^2+2=0$$

take the derivative of $A(S)$ with respect to S to get;

$$\frac{dA(S)}{dS} = \bar{A}(S) = -4S$$

now the row of zeros in the Routh tabulation is replaced by the coefficients of $\bar{A}(S)$ and the new tabulation is:

S^4	1	-3	2
S^3	1	-1	0
S^2	-2	2	0
S^1	-4	0	
S^0	2		

and the system is unstable.

Method (2):

By dividing the characteristic equation by the auxillinary equation to obtain the reduced-order polynominal.

Example: Consider the equation:

$$S^6 + 6S^5 + 10S^4 + 12S^3 + 13S^2 - 18S - 24 = 0$$

Solution:

S^6	1	10	13	-24
S^5	6	12	-18	0
S^4	8	16	-24	0
S^3	0	0	0	0
S^2				

the auxiliary equation $A(S)$ is obtained from the coefficients of the preceding row as follows:

$$A(S) = 8S^4 + 16S^2 - 24$$
$$= 8(S^4 + 2S^2 - 3)$$
$$= 8(S^2 - 1)(S^2 + 3)$$

dividing the polynomial by the known factor $(A(S) = S^4 + 2S^2 - 3)$ yields:

$$\begin{array}{r|l} & S^2 + 6S + 8 \\ \hline S^4 + 2S^2 - 3 & S^6 + 6S^5 + 10S^4 + 12S^3 + 13S^2 - 18S - 24 \\ & \mp S^6 \quad \mp 2S^4 \quad \mp 3S^2 \\ \hline & 6S^5 + 8S^4 + 12S^3 + 16S^2 - 18S - 24 \\ & \mp 6S^5 \quad \mp 12S^3 \quad \mp 18S \\ \hline & 8S^4 + 16S^2 - 24 \\ & \mp 8S^4 \quad \mp 16S^2 \quad \mp 24 \\ \hline & 0 \end{array}$$

Thus, the original characteristic equation may be written in the form :-

$$S^6 + 6S^5 + 10S^4 + 12S^3 + 13S^2 - 18S - 24 = 0$$

$$(S^4 + 2S^2 - 3)(S^2 + 6S + 8) = 0$$

$$(S^2 - 1)(S^2 + 3)(S + 2)(S + 4) = 0$$

therefore, the roots are :

$S_{1,2} = \mp 1$ $S_{3,4} = \mp\sqrt{3}j$ $S_5 = -2$ $S_6 = -4$

So, the system is unstable.

Example: The characteristic equation for certain feed-back control system is given below. Determine the range of (K) that correspond to a stable system.

$$S^3 + 1040S^2 + 48500S + (4\times10^5)K = 0$$

Solution:

S^3	1	48500
S^2	1040	$4\times10^5 K$
S^1	$\frac{50440000 - 4\times10^5 K}{1040}$	0
S^0	$4\times10^5 K$	

for this system to be stable, all the coefficients in the first column of the Routh tabulation must have the same sign. This lead to the following condition :

$$\frac{5044\times10^4 - 4\times10^5 K}{1040} > 0 \Rightarrow K < 126.1$$

or

$4\times10^5 K > 0 \quad \Rightarrow \quad K > 0$

As a result, the condition of asymtotic stability of the overall system is :

$$0 < K < 126.1$$

Example: Consider the equation: $S^3 + 3KS^2 + (K+2)S + 4 = 0$. Determine the range of K, so that the system is stable.

Solution:

S^3	1	$K+2$
S^2	$3K$	4
S^1	$\frac{3K(K+2)-4}{3K}$	0
S^0	4	

from the S^2 row $\Rightarrow 3K > 0 \quad \therefore K > 0$

from the S^1 row $\Rightarrow \frac{3K(K+2)-4}{3K} > 0 \Rightarrow 3K^2 + 6K - 4 \geq 0$

$\therefore K > -2.528 \quad$ or $\quad K > 0.528$

for the closed-loop system to be stable K must satisfy,

$K > 0.528$

Example: Consider the equation: $S^3 + (K+0.5)S^2 + 5KS + 50 = 0$

Solution:

S^3	1	$5K$
S^2	$(K+0.5)$	50

$$
\begin{array}{c|cc}
S^1 & \dfrac{5K^2+2.5K-50}{K+0.5} & 0 \\
S^0 & 50 &
\end{array}
$$

$K+0.5>0 \quad \Rightarrow \quad K>-0.5$

$5K^2+2.5K-50>0 \quad \Rightarrow \quad K> \dfrac{-2.5\mp\sqrt{2.5^2+(4\times5\times50)}}{2\times5}$

$\Rightarrow K>-3.42 \quad \text{or} \quad K>2.92$

So, $K>2.92$ for stable system.

Example: The open-loop transfer function of a unity feedback system is given by:

$$G(S)=\frac{K}{S(S+3)(S^2+S+1)}$$

Determine the values of K that will cause sustained oscillation in the closed loop system. Also, find oscillation frequency.

Solution:

The characteristic equation is:

$$1+G(S)H(S)=1+\frac{K}{S(S+3)(S^2+S+1)}=0$$

or

$$S(S+3)(S^2+S+1)+K=0$$

$$S^4+4S^3+4S^2+3S+K=0$$

$$
\begin{array}{c|ccc}
S^4 & 1 & 4 & K \\
S^3 & 4 & 3 & 0 \\
S^2 & 13/4 & K & 0 \\
S^1 & \dfrac{(39/4)-4K}{13/4} & 0 & 0 \\
S^0 & & &
\end{array}
$$

The condition for system stability is:

$$K > 0$$

$$\frac{39}{4} - 4K > 0 \quad \Rightarrow \quad K < \frac{39}{16}$$

therefore the stability $\Rightarrow \frac{39}{16} > K > 0$

when $K = \frac{39}{16}$ there will be a zero at the first entry in the fourth row. This will indicate precence of symmatrical roots.

$\therefore K = \frac{39}{16}$ will cause sustained oscillations, the auxiliary equ.

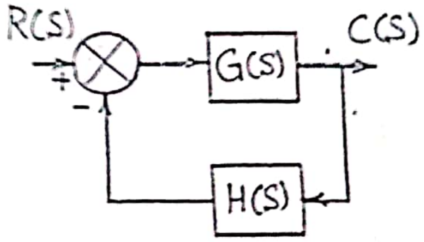
$$\frac{13}{4} S^2 + \frac{39}{16} = 0 \quad \Rightarrow S = \mp\sqrt{0.75}\,j \ ; \ \text{frequency } \omega = 0.86$$

6.1 Introduction to Root-Locus Method.

The basic characteristic of the transient response of a closed-loop system is closely related to the location of the closed-loop poles. If the system has a variable loop gain, then the location of the closed-loop poles depends on the value of the loop gain chosen. It is important, therefore, that the designer know how the closed-loop poles move in the S plane as the loop gain is varied.

The closed-loop poles are the roots of the characteristic equation. A simple method for finding the roots of the characteristic equation has been developed by W.R. Evans and used extensively in control engineering. This method, called the "root-locus method", is one in which the roots of the characteristic equation are plotted for all values of a system parameter. The roots corresponding to a particular value of this parameter can then be located on the resulting graph.

For the negative feedback system shown in the following fig., the transfer function is



$$\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)}$$

the characteristic equation is

$$1+G(S)H(S)=0 \quad \text{or} \quad G(S)H(S)=-1$$

this equation can be split into two equations by equating the angles and magnitudes of both sides, respectively, to obtain the following:

Angle condition

$$\angle G(S)H(S) = \pm 180(2K+1) \qquad K=0,1,2,\dots$$

Magnitude condition:

$$|G(S)H(S)| = 1$$

The values of S that fulfill both the angle and magnitude conditions are the roots of the characteristic equation, or the closed loop poles. A locus of the points in the complex plane satisfying the angle condition alone is the root locus. The roots of the characteristic equation (the closed-loop poles) corresponding to a given value of the gain can be determined from the magnitude condition.

In many cases, G(S) H(S) involves a gain parameter K, and characteristic equation may be written as

$$1 + \frac{K(S+Z_1)(S+Z_2)\cdots(S+Z_m)}{(S+P_1)(S+P_2)\cdots(S+P_n)} = 0$$

or

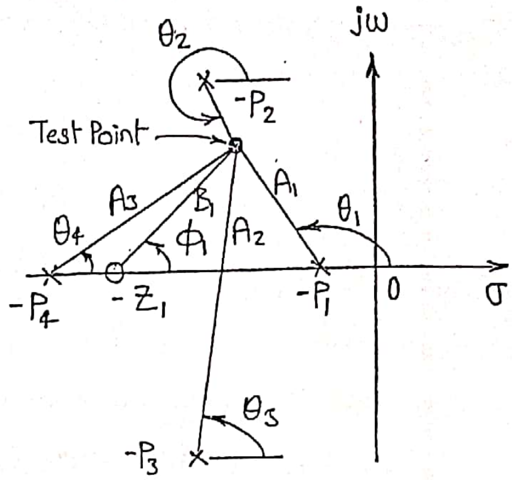
$$(S+P_1)(S+P_2)\cdots(S+P_n) + K(S+Z_1)(S+Z_2)\cdots(S+Z_m) = 0$$

Then the root loci for the system are the loci of the closed-loop poles as the gain K is varied from zero to infinity.

Note that to begin sketching the root loci of a system by the root-locus method we must know the location of the poles and zeros of G(S) H(S). Remember that the angles of the complex quantities originating from the open-loop poles and open-loop zeros to the test point S are measured in the counterclockwise direction. For example, if G(S) H(S) is given by

$$G(S)H(S) = \frac{K(S+Z_1)}{(S+P_1)(S+P_2)(S+P_3)(S+P_4)}$$

where $-P_2$ and $-P_3$ are complex-conjugate poles, then the angle of $G(S)H(S)$ is



$$\underline{/G(S)H(S)} = \phi_1 - \theta_1 - \theta_2 - \theta_3 - \theta_4$$

where $\phi_1, \theta_1, \theta_2, \theta_3, \theta_4$ are measured counterclockwise as shown in the figure.

<u>OR</u> the angle condition can be written as;

$$\measuredangle(S+Z_1) - \measuredangle(S+P_1) - (S+P_2) - \measuredangle(S+P_3) - \measuredangle(S+P_4) = \mp i\pi$$

the magnitude of $G(S)H(S)$ for this system

$$|G(S)H(S)| = \frac{KB_1}{A_1 A_2 A_3 A_4}$$

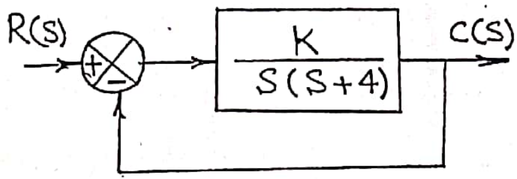
where A_1, A_2, A_3, A_4 and B_1 are the magnitude of the complex quantities $(S+P_1), (S+P_2), (S+P_3), (S+P_4)$ and $(S+Z_1)$ respectively.

<u>OR</u> the magnitude condition can be written as

$$\frac{|S+Z_1|}{|S+P_1||S+P_2||S+P_3||S+P_4|} = \frac{1}{|K|} = \frac{1}{K}$$

The root locus method was originally developed for determining the loci of the roots of the characteristic equation of the single-input, single-output control system as "K" varied from zero to infinity.

Consider the control system represented by the figure:



The characteristic equation for the shown system is:

$$S^2 + 4S + K = 0$$

the roots of this equation are:

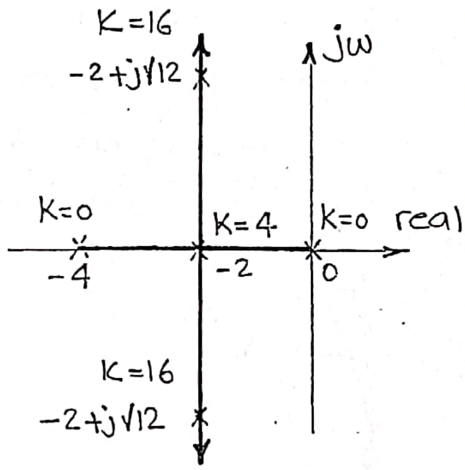
$$r_{1,2} = -2 \mp \sqrt{4-K}$$

these roots can be written as:

$$r_{1,2} = -2 \mp \sqrt{4-K} \qquad K<4$$

$$r_{1,2} = -2 \qquad K=4$$

$$r_{1,2} = -2 \mp j\sqrt{K-4} \qquad K>4$$



When

$K=0 \Rightarrow r_1 = 0 \qquad r_2 = -4$

$K=4 \Rightarrow r_1 = r_2 = -2$

$K=16 \Rightarrow r_1 = -2+j\sqrt{12} \qquad r_2 = -2-j\sqrt{12}$

such that a plot of the roots of the characteristic equation for each value of "K" varied from 0 to ∞ is a root-locus plot as shown.

6.2 The Root-Locus Procedure.

The root-locus is constructed by using certain rules that are now developed in the following. From the general form of the loop of the transfer function:

$$G(S)H(S) = \frac{K(S+Z_1)(S+Z_2)\cdots(S+Z_m)}{(S+P_1)(S+P_2)\cdots(S+P_n)} = -1 \qquad (1)$$

there are polynomials in S of order m and n. It is assumed that $n \geq m$ and the values:

$-Z_1, -Z_2, \cdots, -Z_m$ are the zeros.

and

$-P_1, -P_2, \cdots, -P_n$ are the poles.

We are interested in drawing the loci of the roots of the characteristic equation given in equation (1) as "K" varies from zero to infinity.

Rule 1: Number of Loci

The number of branches of the root loci is equal to the number of the roots given by the order of the characteristic equation. Since it is assumed that $n \geq m$, the characteristic equation has n roots and hence there are n loci.

Rule 2: Origin of Loci

The loci originate when $K=0$ at the poles. When $K=0$, the roots of the characteristic equation are: $-P_1, -P_2, \ldots, -P_n$.

Rule 3: Termination of Loci

When $K \to \infty$, m loci terminate at the m zeros and $(n-m)$ loci terminate at ∞ along asymptotes. When $K \to \infty$, the roots of the characteristic equation are: $-Z_1, -Z_2, \ldots, -Z_m$.

Example: For the characteristic equation:

$$s(S+2)(S+3) + K(S+1) = 0$$

since:

$P_1 = 0,\ P_2 = -2,\ P_3 = -3 \Rightarrow n = 3$ three poles

$Z_1 = -1 \Rightarrow m = 1$ one zero

therefore, there are three branches of the root loci and there are three origins at $S=0$, $S=-2$, $S=-3$ when $K=0$.

on the other hand, when $K \to \infty$ there is one terminate at $S=-1$. So, there are $(n-m) = 2$ two loci go to ∞ along the asymptotes.

Rule 4: Symmetry of the Root Locus.

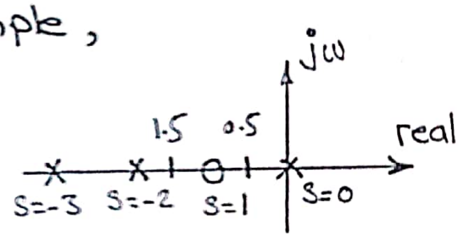
The root locus is symmetry about the real axis of the complex_S plane. Since the values of the parameters of the characteristic equation are real, complex roots always occur in conjugate pairs.

Rule 5: Location of Locus on the Real Axis.

A value of S on the real axis is a point in the root-locus if the total number of poles and zeros on the real ax-is to the right of this point is odd. That means, while con-structing the root_locus on the real axis choose a test poi-nt on it. If the sum of poles and zeros to the right of this point is odd, then this point is a part of the root_loci.

*for example, for the previous example,

$S = -0.5$ is a point of the root_loci because there is one pole right it. While $S = -1.5$ is therefore not a point of the root loci.



Rule 6: Angles of Asymptotes.

As $S \to \infty$, the $(n-m)$ loci that do not terminate at the finite zeros of characteristic equation, approach infinity along asymptotes.

As $S \to \infty$ we can let $\measuredangle(S+Z_i) = \measuredangle S$ for $i = 1, 2, \ldots, m$ and $\measuredangle(S+P_i) = \measuredangle S$ for $i = 1, 2, \ldots, n$. Hence from the angle condition, as $S \to \infty$ we obtain:

$$m(\measuredangle S) - n(\measuredangle S) = \mp i\pi$$

or

$$\measuredangle S = \frac{\mp i\pi}{(n-m)} \qquad i = \begin{cases} 1, 3, 5, \ldots, [(n-m)-1] \\ 1 \quad \text{when } n-m=1 \end{cases}$$

Rule 7: Intersection of Asymptotes.

The point where the asymptotes intersect the real axis is given by:

$$\sigma_c = \frac{\sum \text{Poles} - \sum \text{Zeros}}{n-m}$$

OR

$$\sigma_c = \frac{\sum_{i=1}^{n}(-P_i) - \sum_{i=1}^{m}(-Z_i)}{n-m}$$

for example, from the above example, one can get:

$$n=3 \qquad m=1$$

therefore, the angle of asymptotes:

$$\measuredangle S = \frac{\mp i\pi}{3-1} = \mp 90^\circ$$

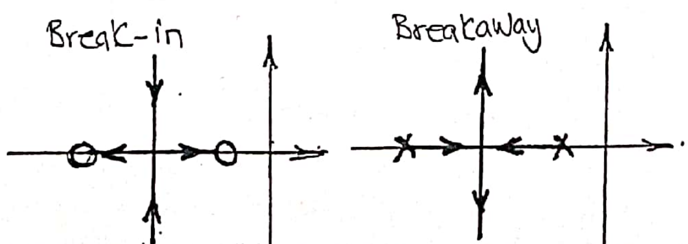
and the intersection with real axis:

$$\sigma_c = \frac{(0-2-3)-(-1)}{3-1} = -2 .$$

Rule 8: Breakaway and Break-in Points.

When two poles on the real axis are connected by a locus, the loci approach each other as K increases until they meet and then depart from the real axis at a point called the breakaway point. The locus can also enter the real axis at a point that is called the break-in point. The location of the breakaway and break-in points are determined from the condition:

$$\frac{dK}{dS} = 0$$



Rule 9: Angles of Departure and Arrival.

The angle of departure of the locus at $K=0$ from a complex pole and the angle of arrival of the locus at $K \to \infty$ at a complex zero determined from the application of the angle condition.

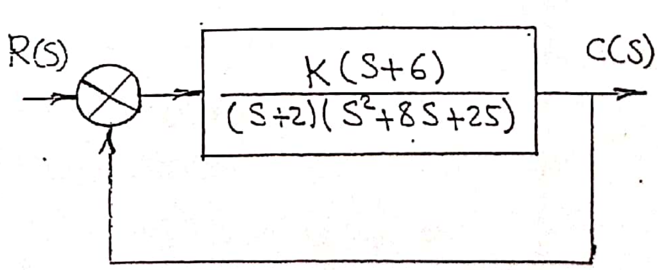
Summary of Procedure.

1. Obtain the characteristic equation in the form:

$$1+\frac{K(S+Z_i)}{(S+P_i)}=0$$

2. Locate the poles as (X) and zeros as (o) in S-plane.
3. Determine the number of loci from Rule 1.
4. Obtain the location of the root locus on the real axis from Rule 5.
5. Determine the angle of asymptotes from Rule 6.
6. Obtain the Intersection of asymptotes with the real axis from Rule 7.
7. Find the breakaway and break-in points (if any) Rule 8.
8. Obtain the angle of departure from complex poles and the angle of arrival at the complex zeros (if any) Rule 9.
9. If the locus crosses the Imaginary axis, then determine the corresponding value of K. The Routh criterion may be employed for this purpose.

Example: In the figure shown here, draw the root locus as K varies from zero to infinity.



Solution:

1. the characteristic equation of the closed-loop system is given by:

$$(S+2)(S^2+8S+25)+K(S+6)=0$$

or

$$\frac{K(S+6)}{(S+2)(S^2+8S+25)}=-1$$

2. It can be observed that there are three poles at:

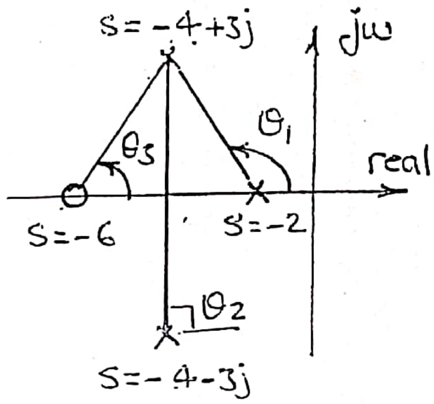
if $K=0 \Rightarrow S=-2,\ S=-4+3j,\ S=-4-3j$

$\therefore n=3$ at $K=0$ (3 poles)

if $K=\infty \Rightarrow S=-6$

$\therefore m=1$ at $K=\infty$ (1 zero)

$\therefore$ there are three loci.



3. The angle of asymptotes is:

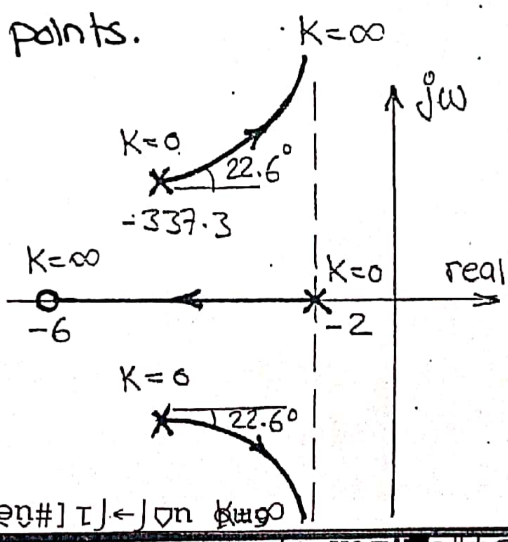
$$\measuredangle S=\frac{\mp i\pi}{n-m}=\mp 90^\circ$$

4. Intersection of asymptotes:

$$\sigma_c=\frac{\sum Poles-\sum Zeros}{n-m}=\frac{(-2-4+3j-4-3j)-(-6)}{3-1}=-2$$

5. There are no breakaway or break-in points.

6. The angle of departure from complex poles at $-4+3j$, we let S be a point on the root-locus infinitesimally close to $-4+3j$ as shown in the fig. We draw vectors to this point from all poles and zeros to obtain:



$$\theta_1=\measuredangle(S+2)=\pi-\tan^{-1}\left(\frac{3}{2}\right)=123.7^\circ$$

$\theta_2 = \measuredangle(S+4+3j) = 90^\circ$

$\theta_3 = \measuredangle(S+6) = \tan^{-1}(\frac{3}{2}) = 56.3^\circ$

Substituting the angle values in the angle condition results in

$$56.3 - 123.7 - \measuredangle(S+4-3j) - 90^\circ = +180$$

$$\therefore \measuredangle(S+4-3j) = -337.3^\circ \text{ or } 22.6^\circ$$

the angle of departure of the locus from complex conjugate pole at $-4-3j$ is -22.6°.

6. It is seen from the last figure that $S=-3$ is a root of the characteristic equation for a value of "K" that can be determined from the magnitude condition.

$$\frac{|S+6|}{|S+2|\,|S+4-3j|\,|S+4+3j|} = \frac{1}{K}$$

at $S=-3$ there is one pole right it so,

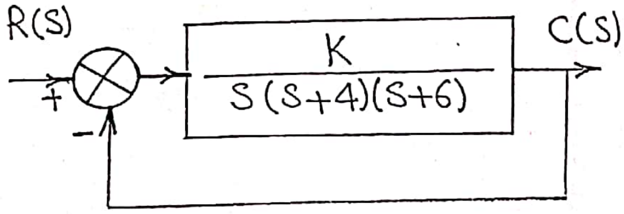
$$\frac{|3|}{|-1|\,|1-3j|\,|1+3j|} = \frac{3}{1 * \sqrt{10} * \sqrt{10}} = \frac{1}{K}$$

$$\therefore K = \frac{10}{3}$$

7. We observe from the root - locus figure that the root locus does not cross the imaginary axis for $0 \leq K < \infty$, Hence, the control system is asymptotically stable for any postive value of K.

Example: Draw the root locus as K varies from zero to Infinity and determine the value of K at which the locus crosses the imaginary axis for the system shown in the figure below.

Solution:



The characteristic equation is:

$$\frac{K}{S(S+4)(S+6)} = -1$$

The angle condition is:

$$\measuredangle S + \measuredangle (S+4) + \measuredangle (S+6) = \mp i\pi$$

and the magnitude condition:

$$\frac{1}{|S|\,|S+4|\,|S+6|} = \frac{1}{K}$$

There are no zeros $\Rightarrow$ $m=0$ at $K=\infty$

The poles are at $S=0$ $S=-4$ $S=-6$ $\Rightarrow$ $n=3$ at $K=0$

The angle of asymptotes are

$$\measuredangle S = \frac{\mp i\pi}{n-m} = \frac{\mp i\pi}{3-0} = \mp 60^\circ$$

$$\sigma_c = \frac{\sum \text{Poles} - \sum \text{zeros}}{n-m}$$

$$= \frac{[0+(-4)+(-6)]-[(0)]}{3-0}$$

$$= -3.33$$

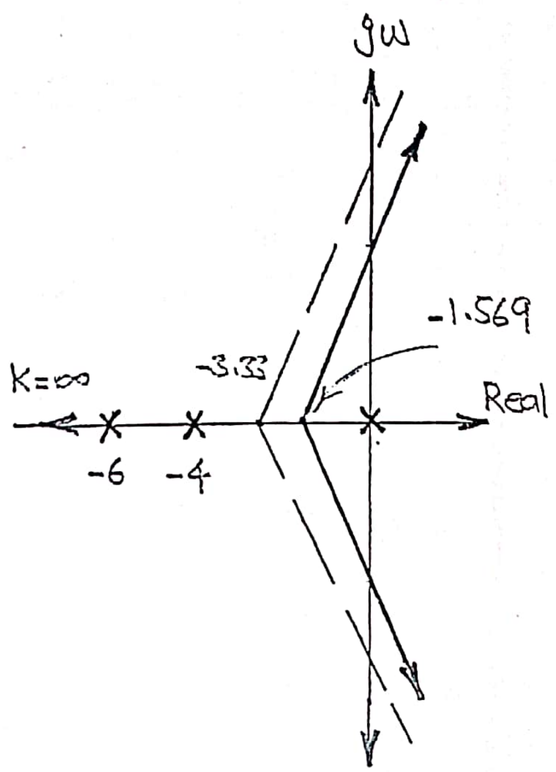
The breakaway point is:

$$S(S+4)(S+6)+K=0 \quad \Rightarrow K = -[S(S+4)(S+6)]$$

$$\therefore K = -(S^3+10S^2+24S)$$

$$\therefore \frac{dK}{dS} = -(3S^2+20S+24) = 0$$

$\therefore$ $S = -1.569$ or $S = -5.097$



The value of K may be computed by Routh-method for the characteristic equation :-

$$S^3 + 10S^2 + 24S + K = 0$$

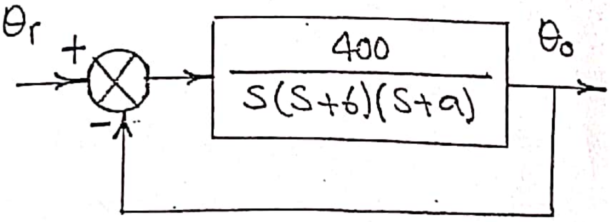
S^3	1	24
S^2	10	K
S^1	$\frac{240-K}{10}$	0
S^0	K	

from S row : $\frac{240-K}{10} = 0 \Rightarrow 240 - K = 0$

$\therefore$ $K = 240$ at the locus cuts the imaginary axis at

$$\omega = \sqrt{24}$$

Example: Draw the root locus for the system shown in the figure below as the parameter a is varied from zero to infinity.



Solution:

The characteristic equation of this system is:

$$S(S+6)(S+a)+400=0$$

which can be re-arranged as:

$$-(S+a)=\frac{400}{S(S+6)} \Rightarrow -a=\frac{400}{S(S+6)}+S$$

$$\Rightarrow -a=\frac{S^3+6S^2+400}{S(S+6)}$$

$$\therefore \quad S^3+6S^2+400+aS(S+6)=0$$

$$1+\frac{aS(S+6)}{S^3+6S^2+400}=0$$

$$\therefore \quad \frac{aS(S+6)}{(S+10)(S-2-6j)(S-2+6j)}=-1$$

The angle condition is:

$$\measuredangle S+\measuredangle(S+6)-\measuredangle(S+10)-\measuredangle(S-2-6j)-\measuredangle(S-2+6j)=\mp i\pi$$

and the magnitude condition is:

$$\frac{|S|\,|S+6|}{|S+10|\,|S-2-6j|\,|S-2+6j|}=\frac{1}{a}$$

$\therefore$ The poles are at: $S=-10$, $S=2+6j$, $S=2-6j \Rightarrow n=3$

The zeros at: $S=0$, $S=-6$ $(m=2)$

The angle of asymptotes is :

$$\measuredangle S = \frac{\mp i\pi}{n-m} = \frac{\mp i\pi}{3-2} = \mp 180^\circ$$

So, the asymptotes is the real axis and hence the point of intersection of asymptotes with real axis is not meaningful.

The breakaway and break-in point can be determined as :

$$a = \frac{-(S^3+6S^2+400)}{S(S+6)}$$

$$\Rightarrow \frac{da}{dS} = \frac{-S(S+6)(3S^2+12S)+(S^3+6S^2+400)(2S+6)}{S^2(S+6)^2}$$

$$\therefore\ 2S^4+15S^3+6S^2+728S+2400=0$$

this quartic polynominal has four roots, the only admissible root is the one between 0 and -6, this root is obtained as -3.03 which is the break-in point.

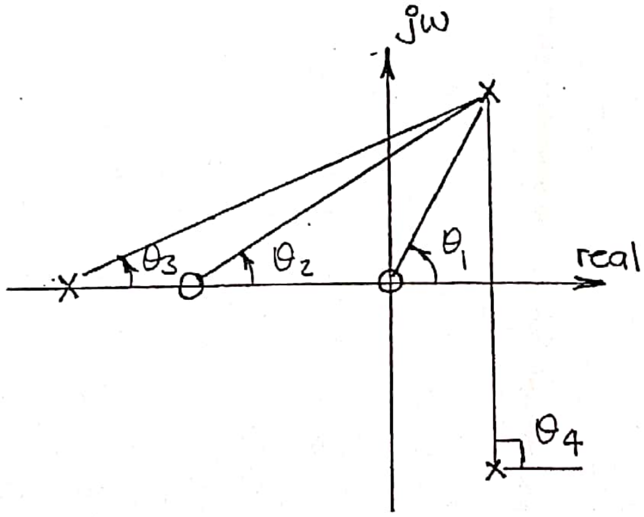
The angle of departure from the complex pole at $S=2+6j$ is obtained from angle condition as :

$$\theta_1 = \measuredangle S = \tan^{-1}\left(\frac{6}{2}\right) = 71.57^\circ$$

$$\theta_2 = \measuredangle(S+6) = \tan^{-1}\left(\frac{6}{8}\right) = 36.87^\circ$$

$$\theta_3 = \measuredangle(S+10) = \tan^{-1}\left(\frac{6}{12}\right) = 26.57^\circ$$

$$\theta_4 = \measuredangle(S-2+6j) = 90^\circ$$



Substituting these values in the angle condition:

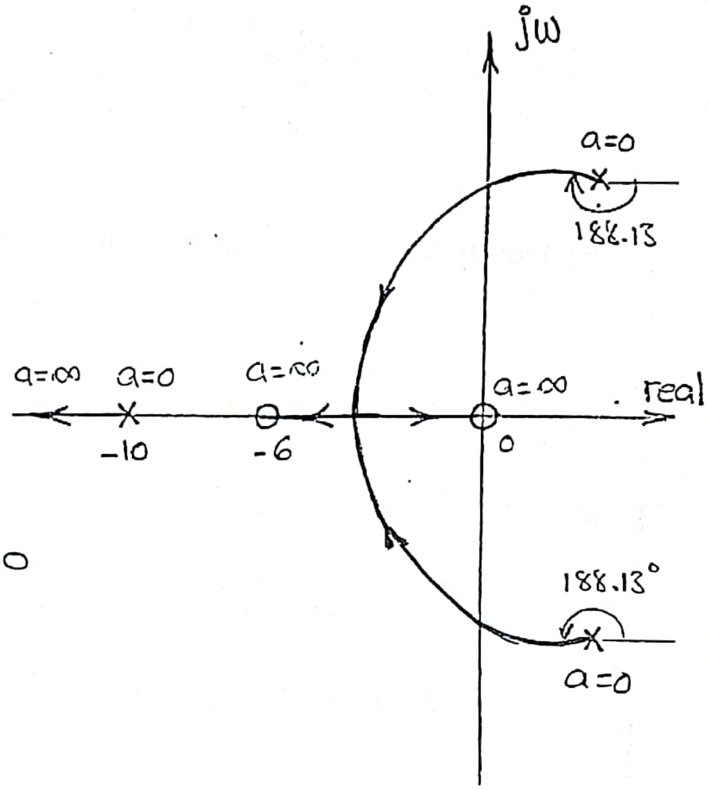
$$71.57° + 36.87° - 26.57° - 90° - \measuredangle(S-2-6j) = 180$$

$$\therefore \measuredangle(S-2-6j) = -188.13°$$

and

$$\measuredangle(S-2+6j) = 188.13°$$

The value of a at which the locus crosses the Imaginary axis can be obtained from the Routh criterion:



$$S^3 + (6+a)S^2 + 6aS + 400 = 0$$

S^3	1	$6a$
S^2	$6+a$	400
S^1	$\dfrac{6a^2+36a-400}{6+a}$	0
S^0	400	

$$\frac{6a^2+36a-400}{6+a} = 0 \qquad \Rightarrow a = 5.7$$

hence, for asymptotes stability of the system we need $a > 5.7$

Example: Determine the root locus for the characteristic equation:

$$1+\frac{K(S+6)}{S(S+4)}=0$$

Solution:

The poles are at: $S=0$, $S=-4$ $\Rightarrow$ $n=2$ at $K=0$

The zeros are at: $S=-6$ $\Rightarrow$ $m=1$ at $K=\infty$

The angle of asymptotes:

$$\measuredangle S=\frac{\mp i\pi}{n-m}=\frac{\mp i\pi}{2-1}=\mp 180^\circ$$

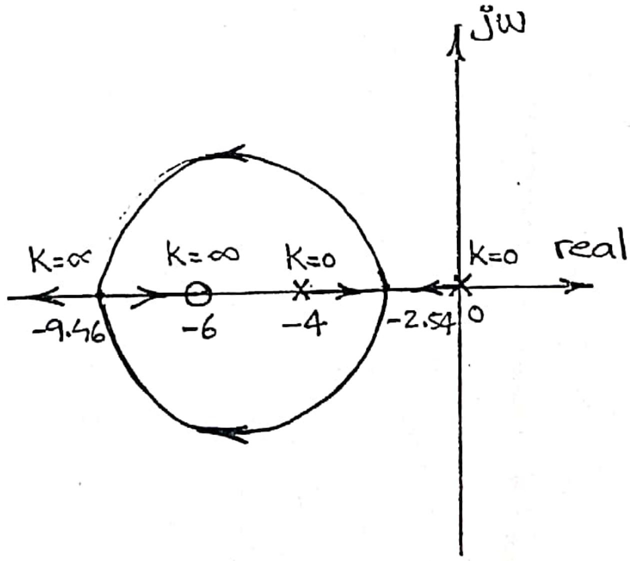
$\therefore$ the asymptotes is the real axis, and hence the points of intersection of the asymptotes with the real axis is not meaningful.

The breakaway and break-in points are:

$$K=\frac{-S(S+4)}{S+6}$$

$$\Rightarrow \frac{dK}{dS}=\frac{-(S+6)(2S+4)+S(S+4)}{(S+6)^2}=0$$

$$\therefore S^2+12S+24=0$$



which gives:

$S=-2.54$ as breakaway point

and

$S=-9.46$ as breakin point.

Basic concepts of measurements

The process or the act of measurement consists of obtaining a quantitative comparison between a predefined A Measurement is an act of assigning a specific value to a physical variable. That physical variable becomes the Measured Variable.

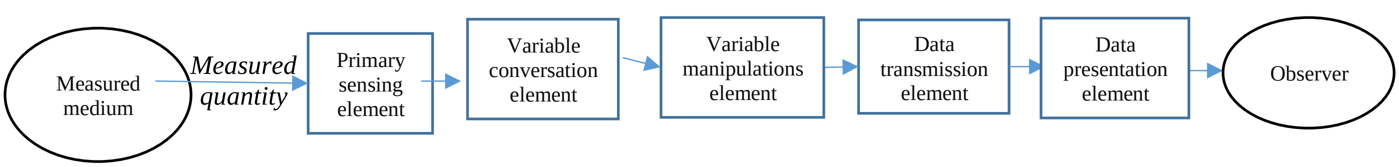
Measurement is also a fundamental element of any control process. The engineer is not only interested in the measurement of physical variables but also concerned with their control. The two function are closely related, however because one must be able to measure a variable such as temperature or flow in order to control it.

Most measurement system may consist of part or all of four general stages:

- A sensor – Transducer Stage.
- An Intermediate Stage or signal – Conditioning Stage.
- A Terminating Stage – Output Stage.
- Feedback – Control Stage.

System configuration:

Instrumentation is used for indicating, measuring and recording physical quantities such as flow, temperature, level, distance, angle, or pressure. The most important function that they perform is to convert data into information. The primary elements of instruments are sensors and transducers. Every instrumentation system contains one or more of the following elements, which represent the possible arrangement of functional element is necessary to describe any instrument.



Measured quantity: is a physical quantity to be measured such as pressure, level, strain, displacement, temperature, etc.

Primary sensing element: it receives energy from the measured medium and produce an output depending.

Variable conversation element: it uses to perform the desired function, which is necessary to convert the measured variable to be more suitable variable.

Variable manipulation element: it uses to change the numerical value according to some definite rule.

Data transmission element: it is necessary to transmit the data from separated physical element to another.

Data presentation element: it is important to recognize the measured quantity by one of the human senses, in order to monitor, control or analysis purpose such as simple indication pointer moving over a scale or recording of a pin moving over a chart.

Measurement systems

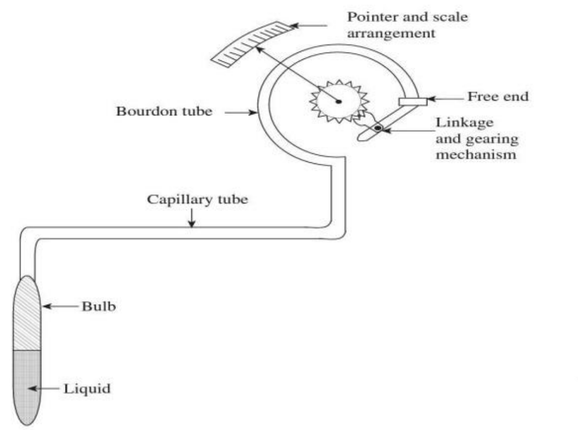
– Choice of instrumentation – Calibration
– Signal Processing and Data acquisition

Different types of transducers

– Measurements with strain gauges
– Pressure transducers
– Position measurements
– Velocity measurements

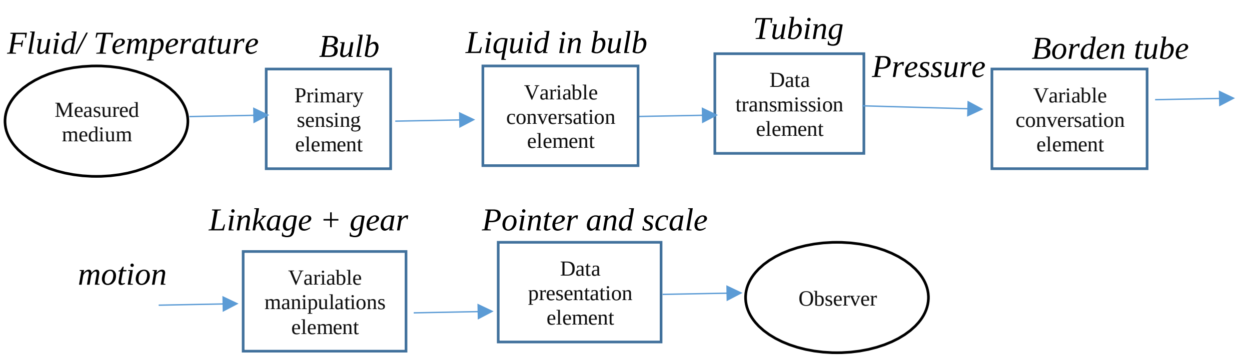
Pressure Thermometer Gauge

This thermometer works on the principle of thermal expansion of the fluid with the change in temperature is to be measured. Temperature change can be determined using these thermometers, which rely on pressure measurement. Usually Mercury is used as liquid Principle of working, where expansion of liquid due to an increase in the pressure within a limited volume Range. It follows the ideal gas law PV = mRT, for constant volume P α T. The change in pressure of the fluid is measured by a suitable pressure transducer, such as the Bourdon tube.

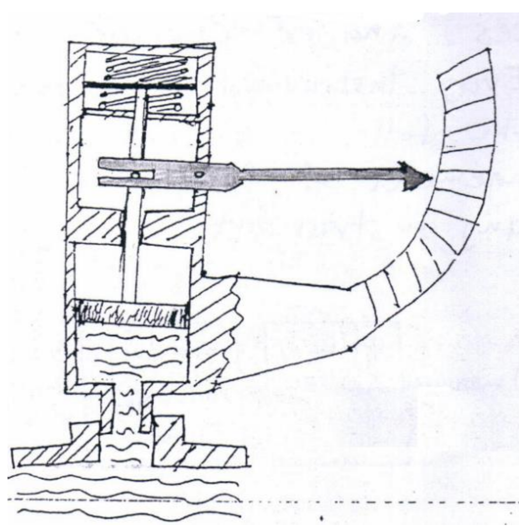


The main constructions of the pressure thermometer (Figure below) are:

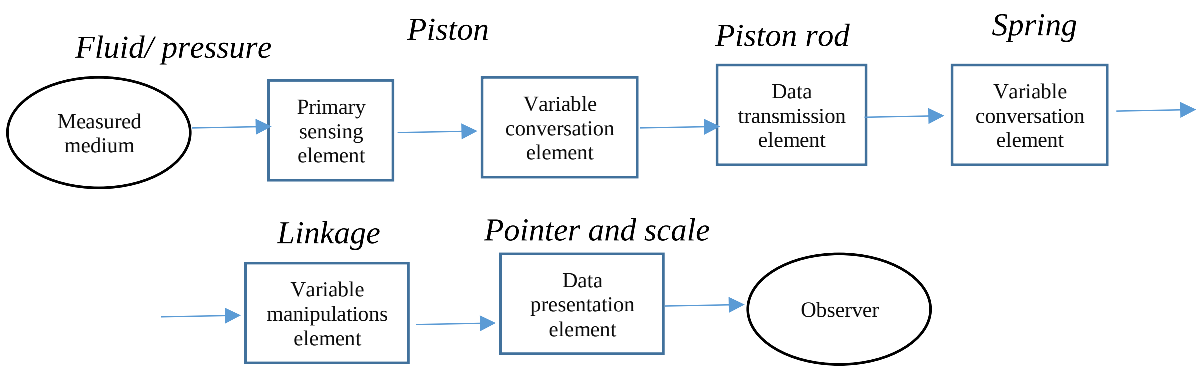
1. Bulb
2. Flexible capillary tube
3. Bourdon tube
4. Linkage and gearing mechanism
5. Pointer and scale arrangement



Pressure Gauge



The primary sensing element is the piston, which also serves the function of the variable conversation. Since, it converts the fluid pressure into an equivalent force on the piston face. The force is transmitted by the piston rod to a spring, which converts forces into a proportional displacement to manipulated by the linkage to give a pointer displacement. The pointer scale indicates the pressure, as presented in data elements shown below:



Accuracy, Error, Precision, and Uncertainty

All measurements of physical quantities are subject to uncertainties in the measurements. Variability in the results of repeated measurements arises because variables that can affect the measurement result are impossible to hold constant. Even if the "circumstances," could be precisely controlled, the result would still have an error associated with it. This is because the scale was manufactured with a certain level of quality, it is often difficult to read the scale perfectly, fractional estimations between scale marking may be made and etc. Of course, steps can be taken to limit the amount of uncertainty but it is always there. Thus, the result of any physical measurement has two essential components:

(1) A numerical value (in a specified system of units) giving the best estimate possible of the quantity measured,

(2) the degree of uncertainty associated with this estimated value.

Definitions

Accuracy of the measurement refers to how close the measured value is to the true or accepted value. If the true value is not known, then the accuracy of measurement can only be estimated (typically, this must be done with extreme care).

Thus,

$$A = \frac{X_n}{Y_n}$$

Where:

A: is the relative accuracy

X_n: is the measured value

Y_n: is the true value

The percentage of accuracy ($\boldsymbol{a}$) is written as:

$$a = \frac{X_n}{Y_n} * 100\%$$

Precision refers to how close together a group of measurements actually are to each other. In many cases, when precision is high and accuracy is low. It is used to indicate the reliability and/or repeatability of a measurement, as reflected by the number of significant figures used to represent the measured value. If the true value is not known, the measured value is repeated multi times so that the precision is a closeness of single measured value with the multi measured values to the same measured variable from the mean of these values.

So,

$$P_i = 1 - \left| \frac{X_i - \overline{X_n}}{\overline{X_n}} \right|$$

Where:

$$\overline{X_n} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

$\boldsymbol{P_i}$: is the precision of measured value of (i)

$\boldsymbol{X_i}$: is the measured value of (i)

$\overline{\boldsymbol{X_n}}$: is the mean value of the multi measured values (n)

Resolution it refers to the ability of instrument to sense the smallest change in the measured variable, which is defined by:

$$Res = \frac{Full\ scale\ deflection}{No.of\ division}$$

Sensitivity it refers to the ratio of the linear movement of the pointer on the instrument to the change of the measured variable, which is defined by:

$$Sens = \frac{Q_{o2} - Q_{o1}}{Q_{i2} - Q_{i1}}$$

Readability it refers to the closeness with which the scale of the instrument may read.

e.g. An instrument is used to measure a parameter X in range from 0 to 50 varying 12 scale, where another instrument is used to measure the same parameter in the same range but having 6 scale. Thus, the first one has higher readability but with less resolution, where:

$$Res\ 1 = \frac{50 - 0}{12} = \frac{50}{12}$$

$$Res\ 2 = \frac{50 - 0}{6} = \frac{50}{6}$$

So, the resolution is decreased with increasing the number of divisions.

Measurement uncertainty: it is a parameter characterizing the range of values within which the value of the measurand can be said to lie within a specified level of confidence. The uncertainty is a quantitative indication of the quality of the result. It is influenced by systematic and random measurement errors. The systematic errors are caused by abnormalities in gain and zero settings of the measuring equipment and tools. The random errors caused by noise and induced voltages and/or currents.

The uncertainty of measuring instruments is usually given by two values: uncertainty of reading and uncertainty over the full scale. These two specifications together determine the total measurement uncertainty.

i. Uncertainty relative to reading

An indication of a percentage deviation without further specification also refers to the reading.

A voltmeter which reads 70,00 V and has a "±5 % reading" specification, will have an uncertainty of 3,5 V (5 % of 70 V) above and below. The actual voltage will be between 66,5 en 73,5 volt.

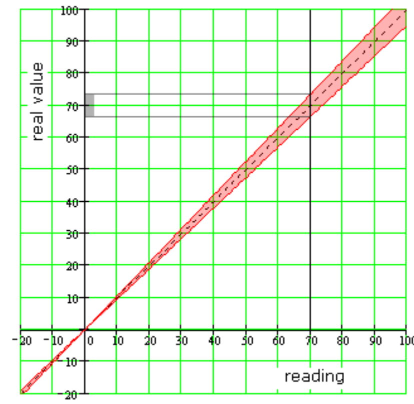


Figure 1 Uncertainty of 5 % reading and a read value of 70 V

ii. Uncertainty relative to full scale

This type of inaccuracy is caused by offset errors and linearity errors of amplifiers. This specification refers to the full-scale range that is used.

A voltmeter may have a specification "3 % full scale". If during a measurement the 100 V range is selected (= full scale), then the uncertainty is 3 % of 100 V = 3 V regardless of the voltage measured. If the readout in this range 70 V, then the real voltage is between 67 and 73 volts.

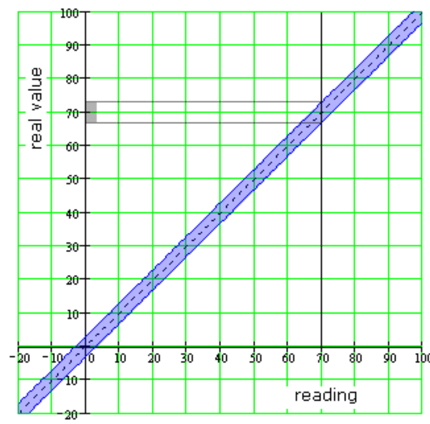


Figure 2 Uncertainty of 3 % full scale in the 100 V range

Figure 2 makes clear that this type of tolerance is independent of the reading. Would a value of 0 V being read; in this case would the voltage in reality between -3 and +3 volts.

<u>Measurement Uncertainty/Error:</u>

The estimated deviation of a measured value from the true value. The true value may or may not be known. There are three types (sources) of error: measurement mistakes, random errors, and systematic errors.

- <u>Measurement mistakes</u> are "illegitimate errors" since they are due to sloppiness and/or lack of care in the measurement process and are avoidable. Mistakes errors should always be completely eliminated.

- Random errors result from (hopefully small) uncontrolled variability of the environment, equipment, and/or other subtle aspects of the measurement. The individual measured values randomly deviate high or low of an average value.

- Systematic errors result in the consistent deviation of a measurement (on average, either high or low as compared to the true value) due to equipment problems or neglect (or ignorance) of some other important factor in the measurement process.

There are three formulae used to express the errors of measurement:

Absolute Error ($\boldsymbol{E_a}$): these errors denote the difference between the true value and the measured value, as:

$$\boldsymbol{E_a = M - T}$$

Where, M is the measurement and T is the true value

Relative Error ($\boldsymbol{E_r}$): it is a relative of the measured quantity to another quantity such as the true value.

$$\boldsymbol{E_r = \frac{E_a}{T} = \left|\frac{M-T}{T}\right|}$$

Percentage Error ($\boldsymbol{E_p}$): If the true value of a quantity is known, the percentage error of a measurement is simply the difference between the measurement M and the true value T, divided by the true value, and then multiplied by 100%.

$$\boldsymbol{E_p = \frac{E_a}{T} * 100\% = \left|\frac{M-T}{T}\right| * 100\%}$$

Classification of errors:

Because errors may arise from every source imaginable, there are many different ways in which they can be classified. Two categories often used to classify the errors, these are:

1.) Systematic errors: This type may be avoided and corrected and can be subdivided into:

   a) Gross Errors: These are mistakes or blunders including:

      i. Misreading of instrument.

ii. Incorrect adjustment of apparatus.
iii. Improper application of instrument.
iv. Computational mistake.

b) Instrument Errors: These are defects or shortcoming of instrument such as:
   i. Error in calibration.
   ii. Damage internal
   iii. Unsuitable internal element.
   iv. Worn and defective parts.

c) Environmental Errors: Physical effects in influence on the: experimental equipment and quantity being measured; these influences are:
   i. Temperature.
   ii. Pressure.
   iii. Humidity.
   iv. Electromagnetic field.

d) Observational Errors: These pertain to habits of the observer, such as:
   i. Imperfect technique.
   ii. Poor judgment.
   iii. Peculiarities in making observations

2.) Random Errors: Random errors are those which are accidental; whose magnitude (and sign) fluctuates in a manner that can't be predicted from a knowledge of the measuring system and the condition of measurement. It can occur for a variety of reasons such as:
   i. Lack of equipment sensitivity.
   ii. Noise in the measurement
   iii. Imprecise definition.

Other Source of Errors:

In addition to the errors mentioned before, there are a number of sources of errors These are:
   i. Noise.
   ii. Response time.
   iii. Design limitation.
   iv. Energy gained or lost by interaction.
   v. Transmission.
   vi. Deterioration of the measuring system.